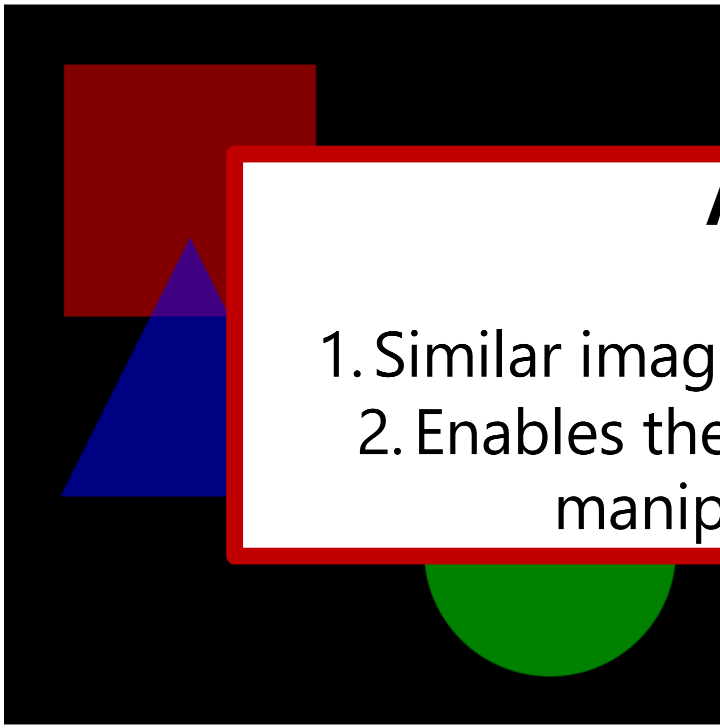
Going from Symbols to Scalars



VSA Formulation

$$I = \mathbf{c}_{red} \odot \mathbf{s}_{square} \odot \mathbf{v}_{top} \odot \mathbf{h}_{left}$$

$$\odot\, \mathbf{h}_{left}$$

$$\odot\, \mathbf{h}_{right}$$

$$\mathbf{h}^{51}$$

$$+\, \mathbf{c}^{465} \odot \mathbf{s}_{triangle} \odot \mathbf{v}^{198} \odot \mathbf{h}^{48.24}$$

$$+\, \mathbf{c}^{532} \odot \mathbf{s}_{circle} \odot \mathbf{v}^{352.1} \odot \mathbf{h}^{344}$$

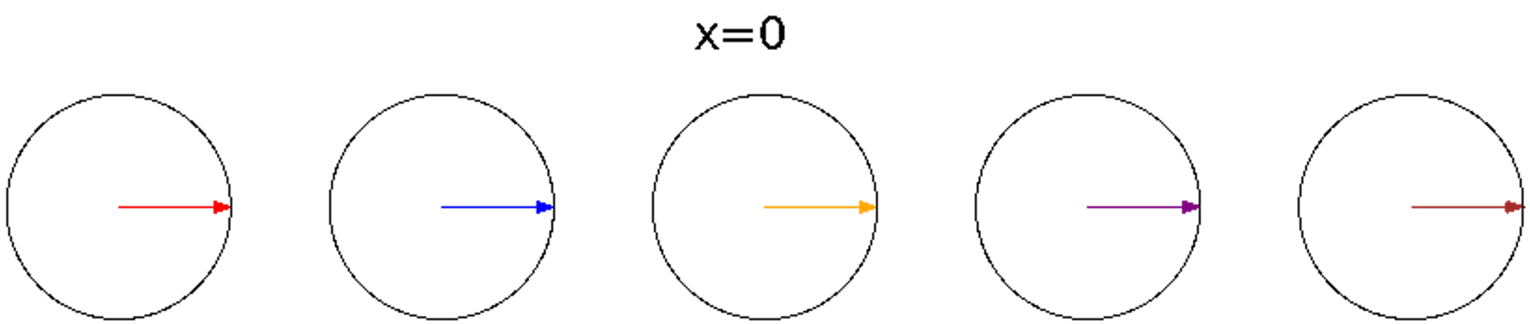
Advantages:

1. Similar images have high inner product
2. Enables the definition, and algebraic manipulation, of functions

Encoding real numbers via **fractional binding**

Key idea 1: Represent any number x, by binding **z** *x* times with itself:

$$\mathbf{z}(x) = \underbrace{\mathbf{z} \odot \cdots \odot \mathbf{z}}_{\text{x times}} = \mathbf{z}^{x}$$



T.A. Plate, “Holographic Recurrent Networks,” Advances in Neural Information Processing Systems (NIPS), pp. 34-41, 1992.
T.A. Plate, “Distributed Representations and Nested Compositional Structure,” University of Toronto, PhD Thesis, 1994.

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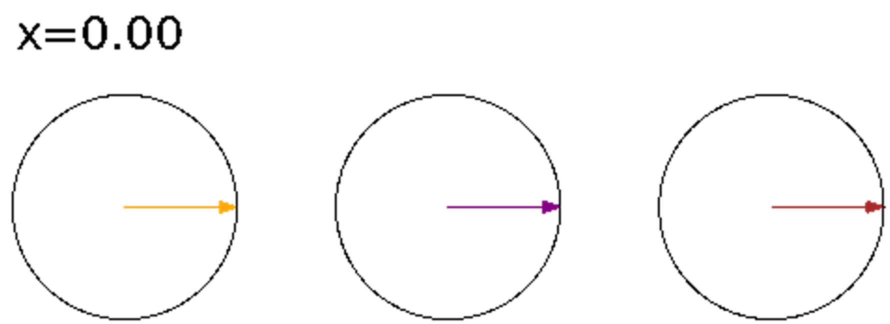
Key idea 2: Extend this definition to support encoding of non-integer *x* values



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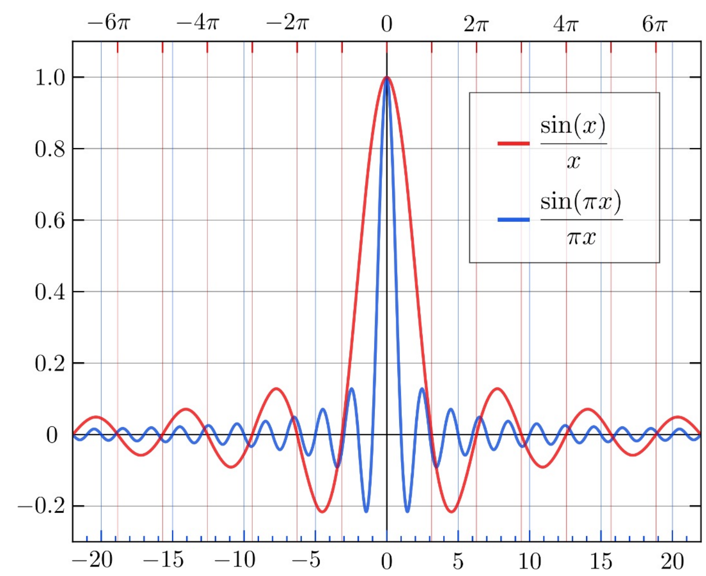
Fractional binding creates a **similarity-preserving encoding (kernel)**

$$z(x)^T \overline{z(x + \Delta x)} = K(\Delta x) \approx \frac{sin(\pi x)}{\pi x}$$

Inner Product **Kernel** **sinc(x)**

K(Δx)



Representing functions with support points

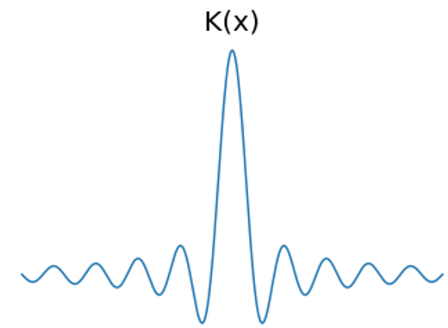
A function:

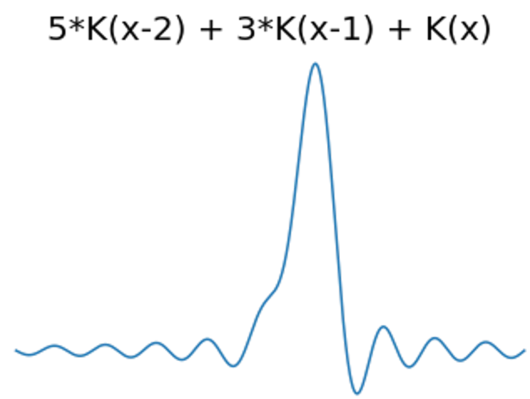
$$f(r) = \sum_k \alpha_k K(r - r_k)$$

is represented as a vector:

$$\mathbf{y}_f \quad = \quad \sum_k \alpha_k \mathbf{z}(r_k)$$

$$f(r) = \mathbf{y}_f^\top \overline{\mathbf{z}(r)}$$





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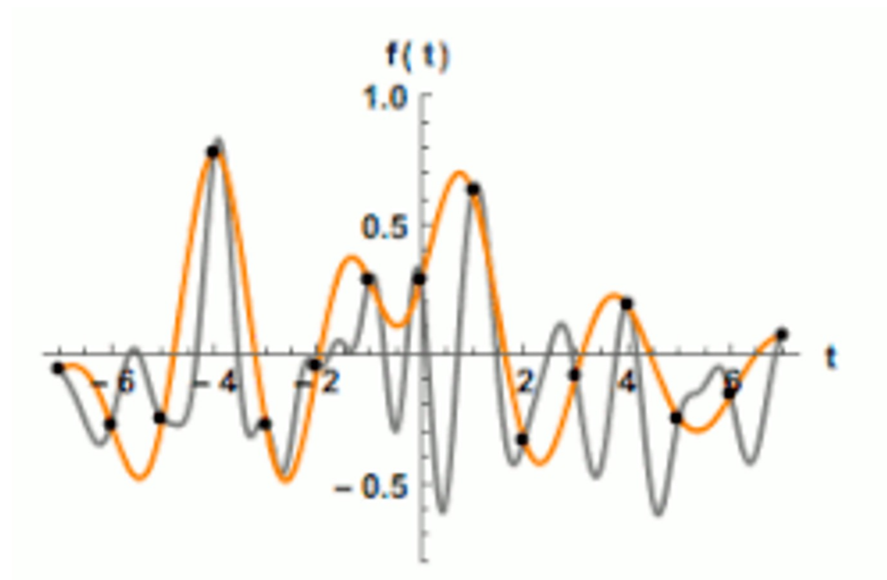
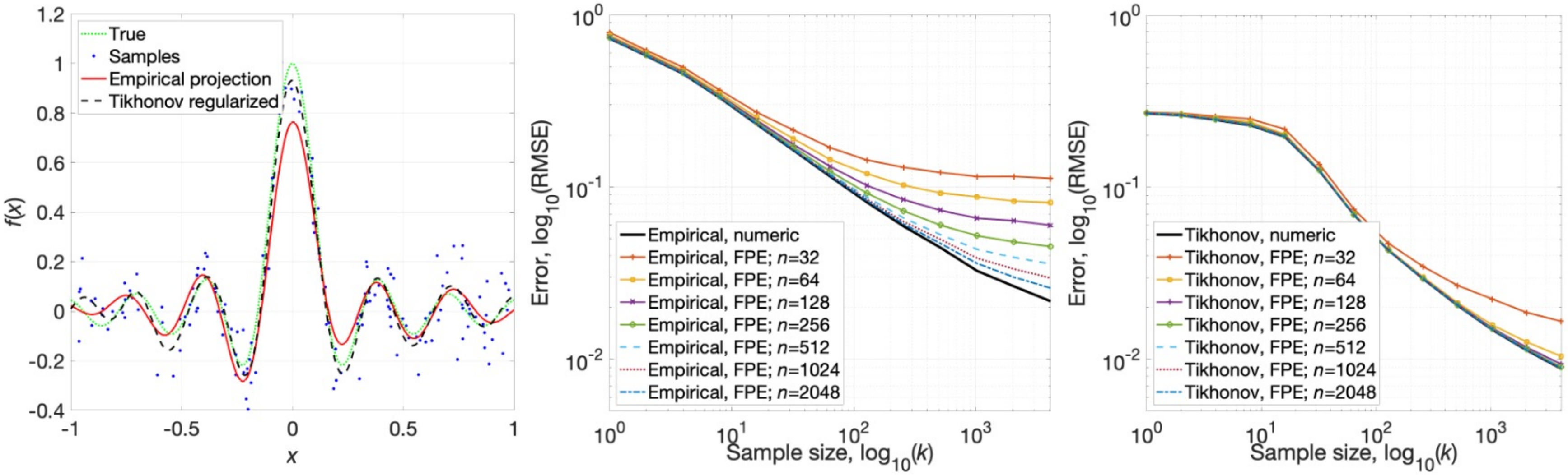


https://en.wikipedia.org/wiki/Whittaker%E2%80%93Shannon_interpolation_formula

Application: regression with sinc kernels

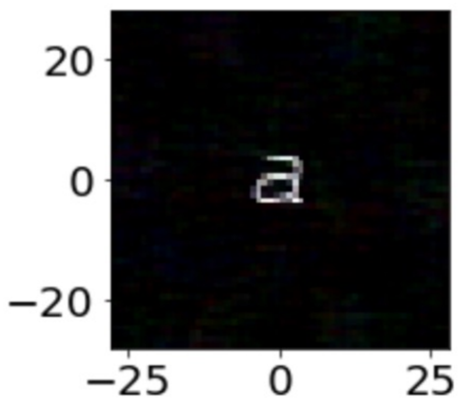
$$\hat{f}_{c,k,n}(x) = \frac{2c}{k\pi}\sum_{i=1}^{k} Y_i \mathrm{sinc}\left(\frac{c}{\pi}(X_i - x)\right) = \frac{2c}{kn\pi}\sum_{i=1}^{k} Y_i \mathbf{z}(\frac{c}{\pi}X_i)^\top \overline{\mathbf{z}(\frac{c}{\pi}x)} = (\mathbf{y}^X)^\top \overline{\mathbf{z}(\frac{c}{\pi}x)},$$



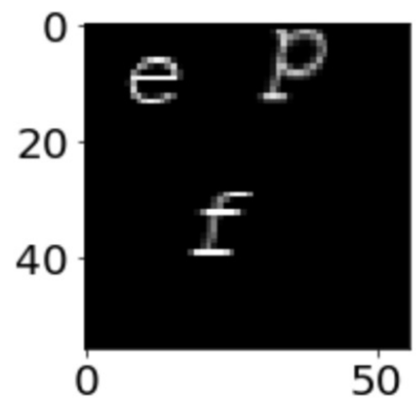
Manipulating functions in VFA

- Point-wise readout of a function $f(s) = \langle f, K_s \rangle = \mathbf{y}_f^\top \overline{\mathbf{z}(s)}$
- Point-wise addition $\mathbf{y}_{f+g} = \mathbf{y}_f + \mathbf{y}_g$
- Function shifting $\mathbf{y}_g = \mathbf{y}_f \circ \mathbf{z}(r)$
  $f(x) \to g(x) = f(x+r)$
- Function convolution $\mathbf{y}_{f*g} = \mathbf{y}_f \circ \mathbf{y}_g$
- Overall similarity between functions $\langle f, g \rangle = \mathbf{y}_f^\top \overline{\mathbf{y}_g}$

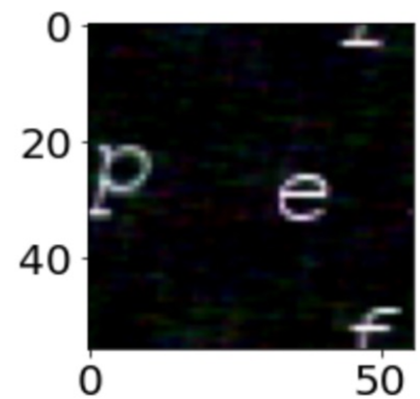
Representing images in VFA



$$\mathbf{v}_a = \sum_{x,y} Im(x, y) \cdot \mathbf{x}^x \odot \mathbf{y}^y$$

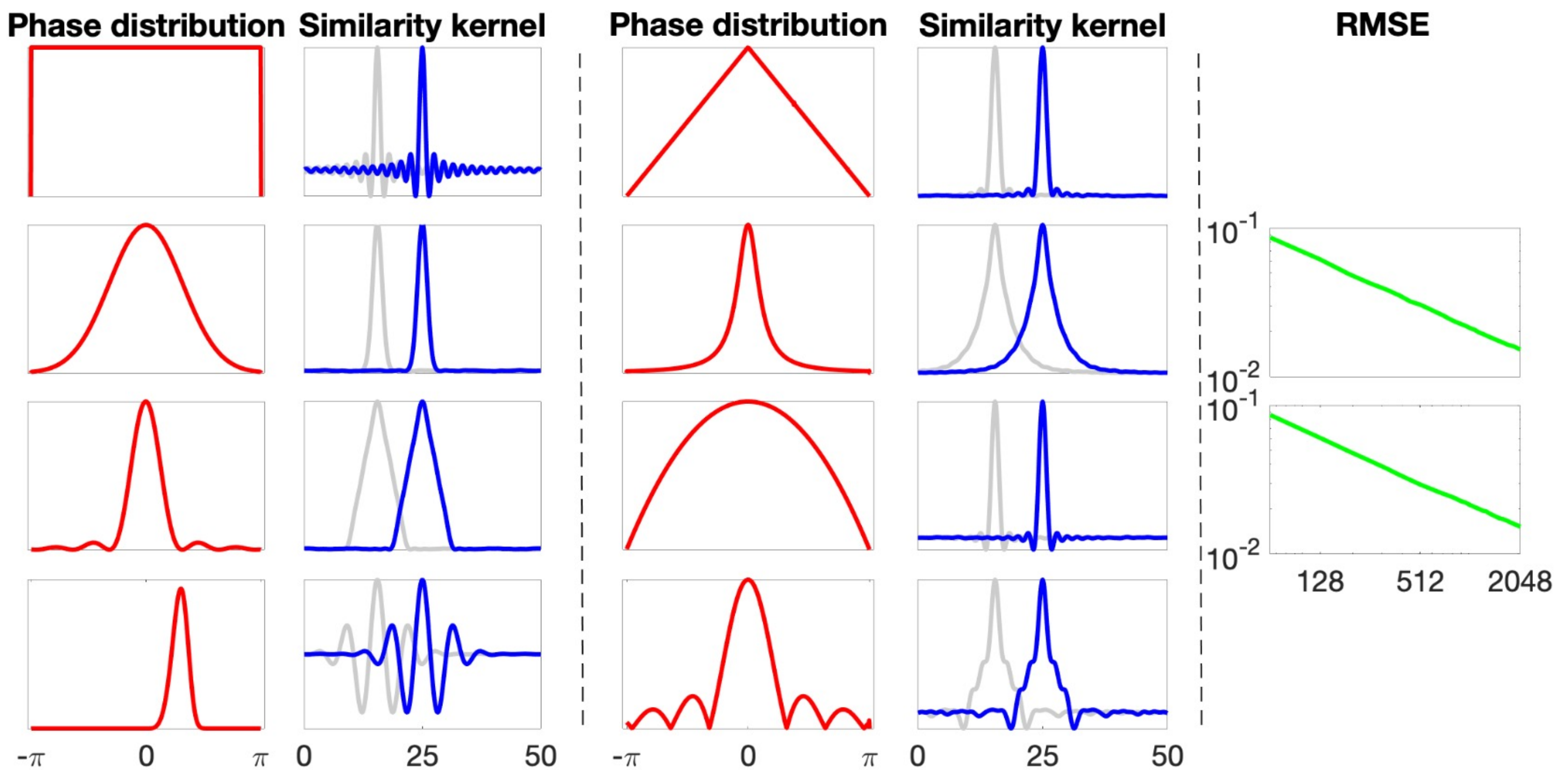


$$\begin{aligned}\mathbf{v}_{scene} &= \mathbf{v}_e \odot \mathbf{x}^{12} \odot \mathbf{y}^{10} \\ &+ \mathbf{v}_p \odot \mathbf{x}^{35.86} \odot \mathbf{y}^{7.22} \\ &+ \mathbf{v}_f \odot \mathbf{x}^{22.7} \odot \mathbf{y}^{35}\end{aligned}$$



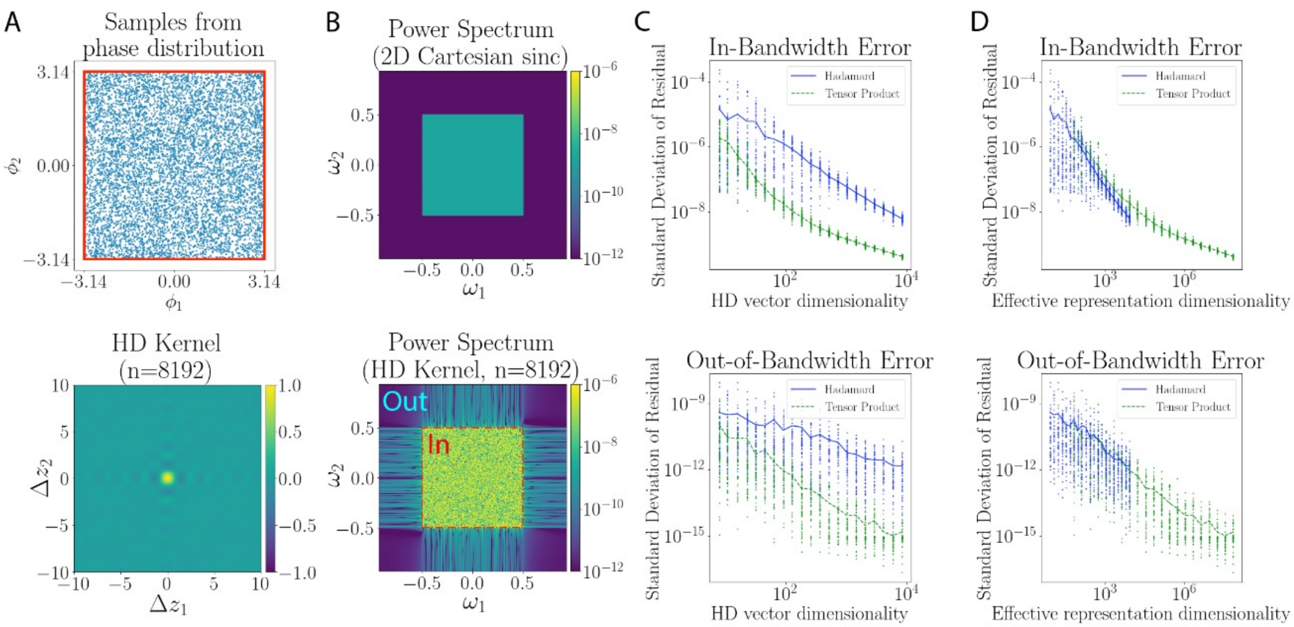
$$\mathbf{v}^{tr}_{scene} = \mathbf{v}_{scene} \odot \mathbf{x}^{25.8} \odot \mathbf{y}^{20.2}$$

Phase distribution of the base vector determines similarity kernel

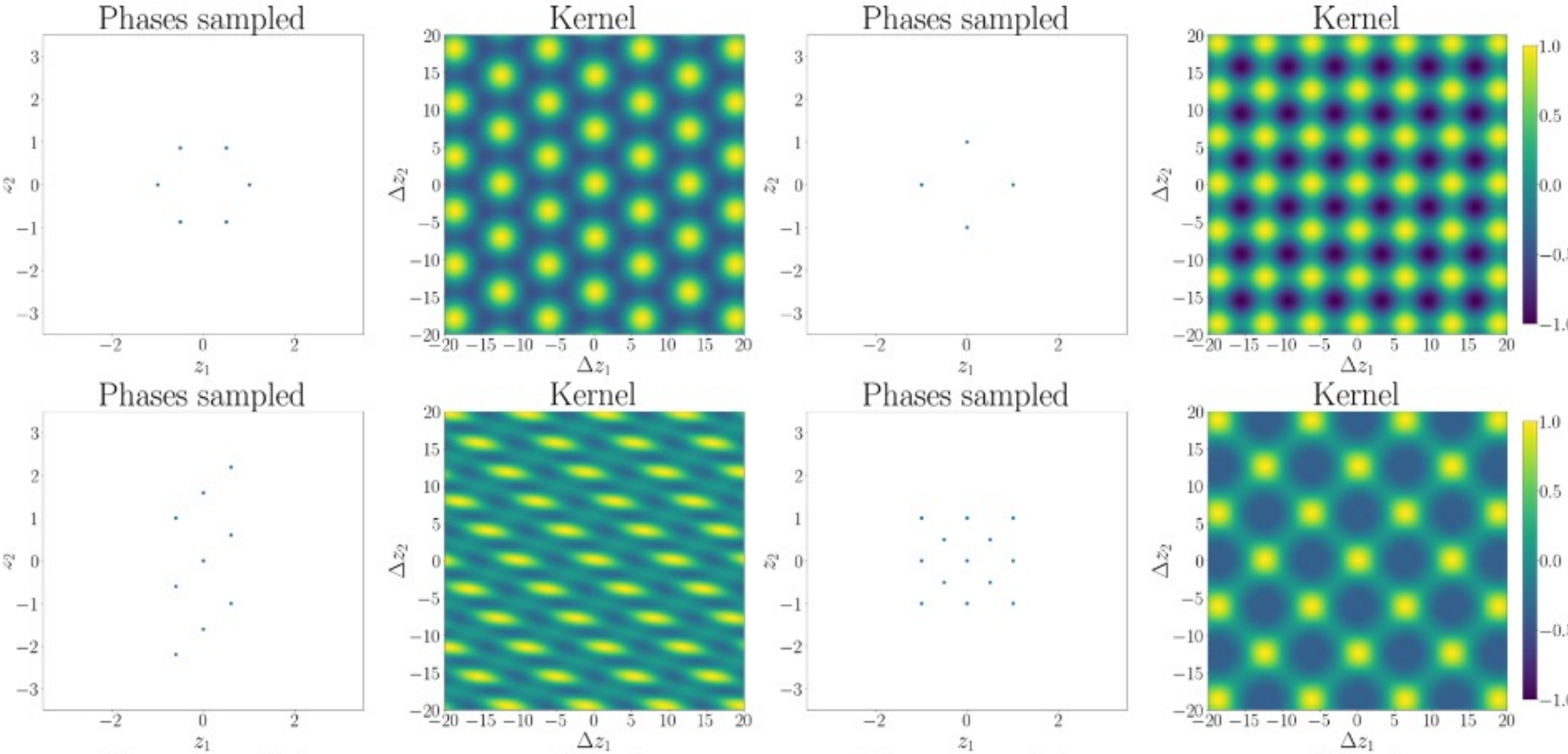


Binding FPEs to produce multi-dimensional kernels

$$\mathbf{z}(\mathbf{r}) = \mathbf{z}_1(r_1) \circ \mathbf{z}_2(r_2) \circ \ldots \circ \mathbf{z}_m(r_m)$$

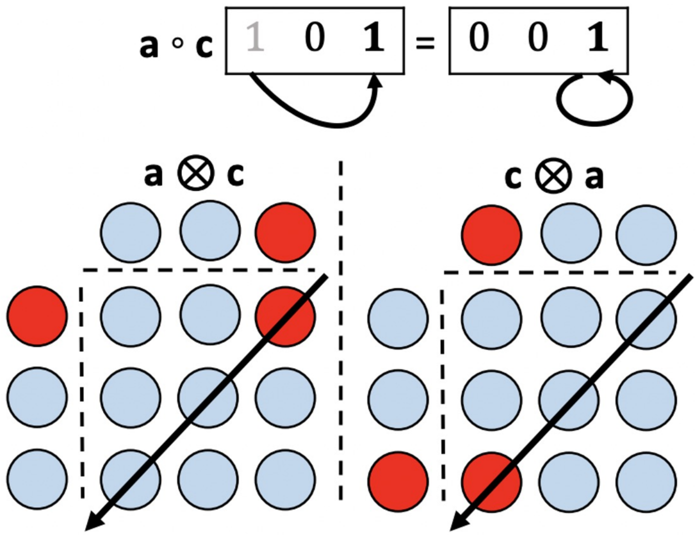


Periodic 2D kernels (Lattices)

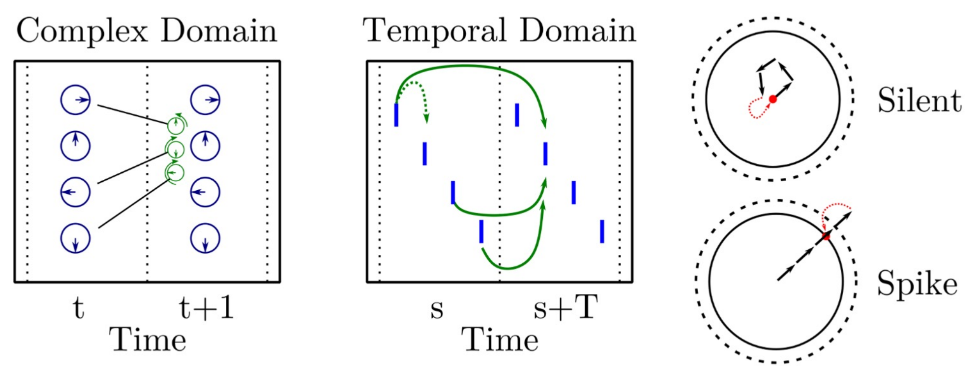


How to map VFA onto neural circuits?

1. Use *sparse* high-dimensional vectors



2. Map amplitudes to firing rate and phases to spike timing



Frady, E.P., Sommer, F.T. (2019) Robust computation with rhythmic spike patterns. PNAS 116(36) 18050-59.
M. Laiho, et al., "High-Dimensional Computing with Sparse Vectors," IEEE Biomedical Circuits and Systems Conference (BioCAS), 2015.
E. P. Frady, et al., "Variable Binding for Sparse Distributed Representations: Theory and Applications," IEEE Transactions on Neural Networks and Learning Systems, 2021.

Sparse Block-Codes

- Due to M. Laiho, et al.
- Seed HD vectors sparse random binary vectors $\{0,1\}^n$
  - n-dimensional HD vector is treated as being constructed from blocks of size k
  - Only one component is active in each block
  - The total number of blocks is n/k
  - Density of HD vector is k/n

Block sparse HD vector: n=18

1 0 0	0 0 1	1 0 0	0 1 0	0 1 0	0 0 1

Block: k=3

- Binding: defined for block sparse HD vectors
  - Cyclic shift within the blocks (but see next slide)
- Superposition: component-wise addition
  - Increases sparsity
  - WTA within the blocks

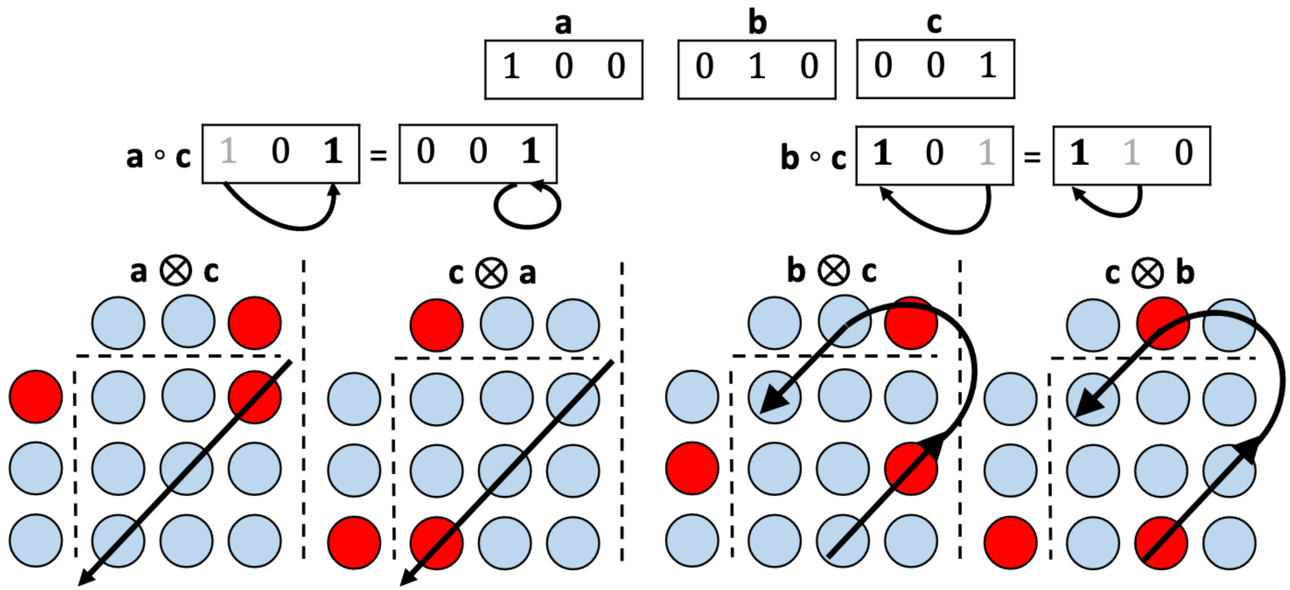
Mika Laiho

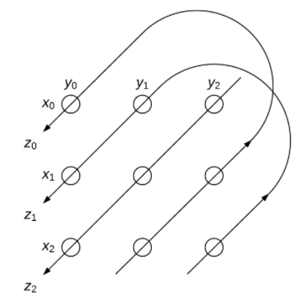
M. Laiho, et al., “High-Dimensional Computing with Sparse Vectors,” IEEE Biomedical Circuits and Systems Conference (BioCAS), 2015.

Slide from Denis Kleyko

Sparse Block-Codes: binding

- Binding: defined for block sparse HD vectors
  - Cyclic shift within the blocks
  - Circular convolution on blocks





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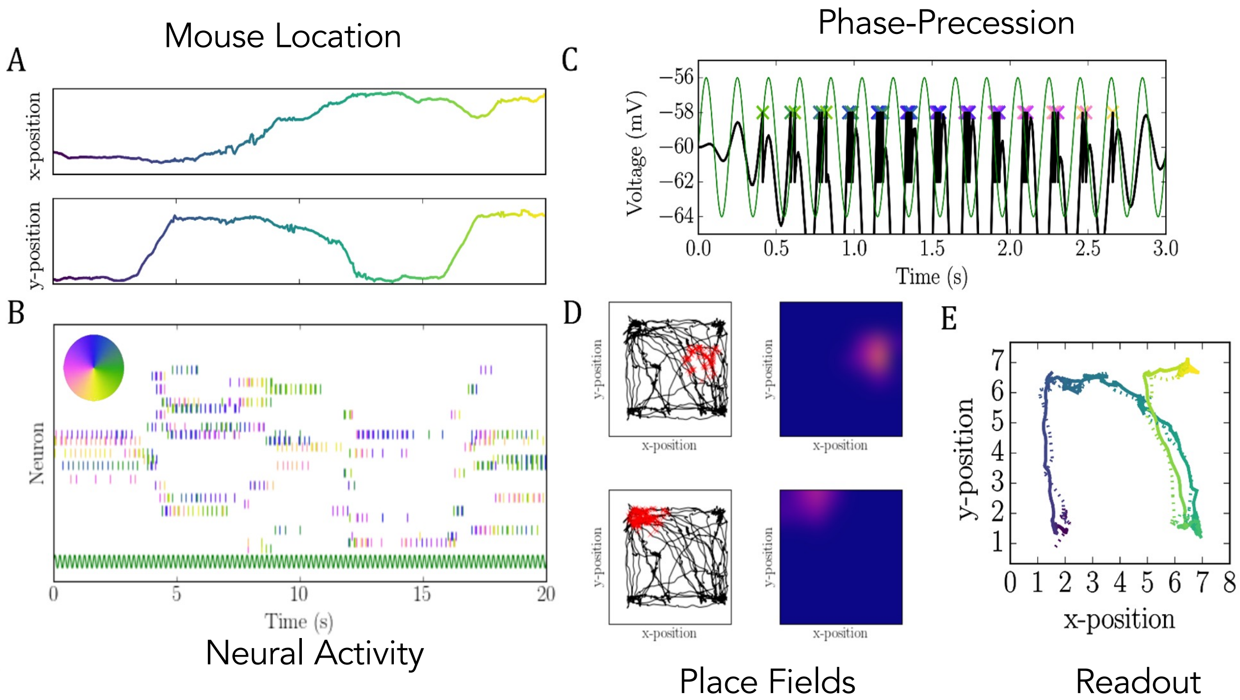
Slide from Denis Kleyko

An encoding model of hippocampus predicts place fields and phase precession



Frady, P., Kanerva, P., & Sommer, F. (2018). A framework for linking computations and rhythm-based timing patterns in neural firing, such as phase precession in hippocampal place cells. In *Proceedings of the Conference on Cognitive Computational Neuroscience*.