Computing with oscillations and waves
One possibility we explore is the case in which the oscillatory trend of the retinal cell does not contain information about the visual stimulus. Even in this situation, the oscillations might increase the amount of information about local retinal features transmitted by the thalamic rate code. They would do so by a process akin to amplitude modulation, in which information about the retinal feature is reproduced in the frequency band of the oscillations. This redundant information could be read out and decoded in the cortex by various mechanisms, such as coincidence detection of afferent inputs or by the relative phase of the thalamic and cortical oscillations. A specific role for the second channel could be de-noising. Further, the amplitude modulation of the afferent spike train generates a signal that might enable cortical oscillations (e.g., relay cells that received periodic synaptic inputs transmitted a significant amount of information in the gamma frequency band. For some cells, the amount of information in the oscillation-based (high frequency) channel was severalfold higher than that conveyed by rate-coded (low frequency) channel; compare Figure 4C with Figure 4D.

Retinal oscillations carry information to cortex (Koepsell, Wang, Hirsch & Sommer 2009)
The Hippocampal Theta Rhythm
(Agarwal, Sommer & Buzsaki 2013)

Foster & Knierim 2012

Buzsaki 2002
Experimental Setup
Oscillatory descriptors

- Real-valued
- Complex-valued
- Demodulated
Demodulation reveals behaviorally relevant signal components
ICA reveals ‘place components’
Coupled oscillator models
Kuramoto model

\[ \dot{\theta}_i = \omega_i + K \sum_j \sin(\theta_j - \theta_i) \]
Oscillation-based Ising machines for solving combinatorial optimization problems
(Wang & Roychowdhury 2019)

\[
\min H \triangleq - \sum_{1 \leq i < j \leq n} J_{ij} s_i s_j - \sum_{i=1}^{n} h_i s_i, \text{ such that } s_i \in \{-1, +1\}
\]
Lyapunov function and dynamics

\[
E(\vec{\phi}(t)) = -K \cdot \sum_{i,j, \, i \neq j} J_{ij} \cdot \cos(\phi_i(t) - \phi_j(t)) - K_s \cdot \sum_{i=1}^{n} \cos(2\phi_i(t))
\]

\[
\frac{d}{dt} \phi_i(t) = -K \cdot \sum_{j=1, \, j \neq i}^{n} J_{ij} \cdot \sin(\phi_i(t) - \phi_j(t)) - K_s \cdot \sin(2\phi_i(t))
\]
Implementation

Fig. 8. A simple oscillator-based Ising machine solving size-8 cubic graph MAX-CUT problems: (a) breadboard implementation with 8 CMOS LC oscillators; (b) illustration of the connections; (c) oscilloscope measurements showing waveforms of oscillator 1~4.

Fig. 9. A size-32 oscillator-based Ising machine: (a) photo of the implementation on perfboards; (b) illustration of the connectivity; (c) a typical histogram of the energy values achieved in 200 runs on a random size-32 Ising problem; the lowest energy level is -88 and is achieved once in this case.
Fig. 10. Coupled oscillators solving MAX-CUT benchmark problem G1 [43] to its best-known cut size 11624.

Graph coloring

(a) Phases of oscillators evolve over time.

(b) Energy function decreases during the process.

(c) The resulting US map colouring scheme.
Threshold phasor associative memory (TPAM)  
(Frady & Sommer 2019)

Energy function

\[ E(z) = -\frac{1}{2} \sum_{ij} W_{ij} z_i z_j^* + \Theta \|z\|_1 \]

\[ z_i = |z_i|e^{j\phi_i} \]

Dynamics

\[ u_i(t) = \sum_j W_{ij} z_j(t) \]

\[ z_i(t + 1) = g(u_i(t), \Theta(t)) := \frac{u_i(t)}{|u_i(t)|} H(|u_i(t)| - \Theta(t)) \]

\[ \Theta(t) = \theta \sum_i |z_i(t)| = \theta |z(t)| \]
Implementation in spiking neurons

Complex Domain

Temporal Domain

Silent

Spike

A Spiking Circuit

B Decoherent

C Coherent

D Spiking TPAM

E Similarity

F Readout

G Capacity

Patterns Stored (M)
Physics successfully implements Lagrange multiplier optimization

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Contributed by Eli Yablonovitch, August 25, 2020 (sent for review July 27, 2020; reviewed by Thomas Kailath and Stanley Osher)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{Maximization of function $f(x, y)$ subject to the constraint $g(x, y) = 0$. At the constrained local optimum, the gradients of $f$ and $g$, namely $\nabla f(x, y)$ and $\nabla g(x, y)$, are parallel.}
\end{figure}
Fermat’s principle of least-time
(Feynman lectures, chapter 26)

Electromagnetic radiation ranges from radio waves to X-rays with only one number (e.g. wave length) varying, light is somewhere in the middle. It is described by quantum theory, but various approximations exist:

a) Long wavelengths — classical theory of electromagnetic waves.

b) Short wavelengths — goes like particles (photons).

Radiation long compared to wave length → energy high compared to photon approach in geometric Optics (Ray, etc.)

If light takes path such that nearly path takes same time to 1st order, then.

If light goes out of one point to another, time on all routes are equal.

In a medium, the speed is \( \frac{c}{n} \) where \( n \) is index of refraction.

\[ c = \text{speed of light} = 3 \times 10^8 \text{ m/sec} \]

\[ \sin \theta_i = n \sin \theta_r \]

**Table 26–1**

<table>
<thead>
<tr>
<th>Angle in air</th>
<th>Angle in water</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
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<td>36</td>
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</table>

reflection

refraction
26–3 Fermat’s principle of least time

Now in the further development of science, we want more than just a formula. First we have an observation, then we have numbers that we measure, then we have a law which summarizes all the numbers. But the real glory of science is that we can find a way of thinking such that the law is evident.

Fig. 26–4. Illustration of Fermat’s principle for refraction.
Fermat’s principle of least time

light decides which is the shortest time, or the extreme one, and chooses that path. But what does it do, how does it find out? Does it smell the nearby paths, and check them against each other? The answer is, yes, it does, in a way.

Take any path and find the time for that path; then make a complex number, or draw a little complex vector, \( \rho e^{i\theta} \), whose angle \( \theta \) is proportional to the time. The number of turns per second is the frequency of the light. Now take another path; it has, for instance, a different time, so the vector for it is turned through a different angle—the angle being always proportional to the time. Take all the available paths and add on a little vector for each one; then the answer is that the chance of arrival of the photon is proportional to the square of the length of the final vector, from the beginning to the end!

Fig. 26–14. The summation of probability amplitudes for many neighboring paths.