Analogical Reasoning with VSAs/HD computing

What **is** an analogy?

"Analogies are partial similarities between different situations that support further inference"
--Dedre Gentner

"...the very blue that fills the whole sky of cognition--analogy is *everything*, or very nearly so, in my view."
--Douglas Hofstadter

What **is** an analogy?

"Spot bit Jane"

"Fido bit John"

"Jane bit Fido"

...an analogy captures *correspondences* between each of these instances

What **is** an analogy?

Propositions

“Spot bit Jane”

“Fido bit John”

“Jane bit Fido”

bite(Jane, Fido)

predicate(agent, object)

What **is** an analogy?

Four (give or take) tasks of *analogical reasoning*.

1. Retrieval
   - Accessing analogous instances from memory--unconscious and fast.
2. Judgement
   - Estimating the *similarity* of different instances based on structural and semantic properties. Conscious and slow.
3. Mapping
   - Finding *structural correspondences* between instances
4. Inference
   - Applying structure from one instance to a generate or explain a novel instance

Gentner (1983). *Cognitive science* | Thagard et al. (1990). *Artificial intelligence* | Forbus et al. (1995). *Cognitive science*
Hummel & Holyoak (1997). *Psychological Review*

Propositional analogies with HRRs

- Encoding depends on the analogical reasoning task
- Example (from Tony Plate):
    - HRRs encodings that reflect human judgements of similarity

Plate, T. (2000). Analogy retrieval and processing with distributed vector representations

Propositional analogies with HRRs

"Spot bit Jane, causing Jane to flee from Spot"

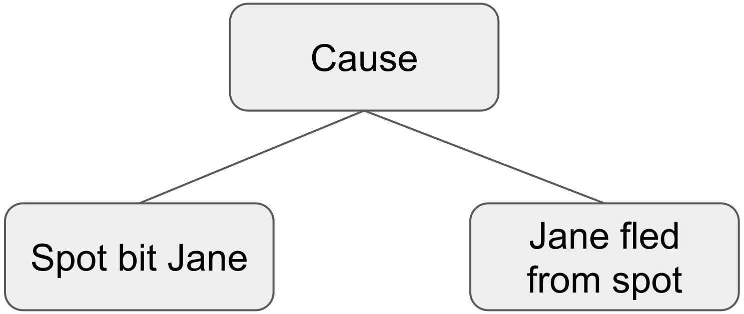
$$\mathbf{P}_{\text{bite}} = \mathbf{bite} + \mathbf{spot} + \mathbf{jane} + \mathbf{bite}_{\text{agt}} \otimes \mathbf{spot} + \mathbf{bite}_{\text{obj}} \otimes \mathbf{jane}$$

$$\mathbf{P}_{\mathbf{flee}} = \mathbf{flee} + \mathbf{spot} + \mathbf{jane} + \mathbf{flee}_{\text{agt}} \otimes \mathbf{jane} + \mathbf{flee}_{\text{from}} \otimes \mathbf{spot}$$

$$\mathbf{P} = \mathbf{cause} + \mathbf{P}_{\mathbf{bite}} + \mathbf{P}_{\mathbf{flee}} + \mathbf{cause}_{\text{antc}} \otimes \mathbf{P}_{\mathbf{bite}} + \mathbf{cause}_{\text{cnsq}} \otimes \mathbf{P}_{\mathbf{flee}}$$

Propositional analogies with HRRs

"Spot bit Jane, causing Jane to flee from Spot"



Propositional analogies with HRRs

Different types of “similarity”

- LS (literal similarity): ‘Fido bit John, causing John to flee from Fido’. (Has both structural and superficial similarity to the probe **P**.)
- SF (surface features): ‘John fled from Fido, causing Fido to bite John’. (Has superficial but not structural

$$\mathrm{LS} > \mathrm{CM} \geq \mathrm{SF} > \mathrm{AN} \geq \mathrm{FOR}$$

superficial similarity, but types of corresponding objects are switched.)

- AN (analogy): ‘Mort bit Felix, causing Felix to flee from Mort’. (Has structural but not superficial similarity.)
- FOR (first-order relations only): ‘Mort fled from Felix, causing Felix to bite Mort’. (Has neither structural nor superficial similarity, other than shared predicates.)

Propositional analogies with HRRs

Table 2

	Episodes in long-term memory	*Commonalities with probe*			*Similarity scores* HRR	MAC
$\mathbf{E}_{\mathbf{LS}}$	Fido bit John, causing John to flee from Fido	✓	✓	✓	0.71	1.0
$\mathbf{E}_{\mathbf{SF}}$	John fled from Fido, causing Fido to bite John	✓	✓	×	0.47	1.0
$\mathbf{E}_{\mathbf{CM}}$	Fred bit Rover, causing Rover to flee from Fred	✓	✓	✓	0.47	1.0
$\mathbf{E}_{\mathbf{AN}}$	Mort bit Felix, causing Felix to flee from Mort	×	✓	✓	0.42	0.6
$\mathbf{E}_{\mathbf{FOR}}$	Mort fled from Felix, causing Felix to bite Mort	×	✓	×	0.30	0.6

P, Spot bit Jane, causing Jane to flee from Spot.

Issues with what we’ve shown so far

- Complicated
- Hand-designed
- How do atomic concepts (vectors) get their meaning?
- Evaluated similarity without further decomposition

Simplifying things

A : B :: C : D

Simplifying things

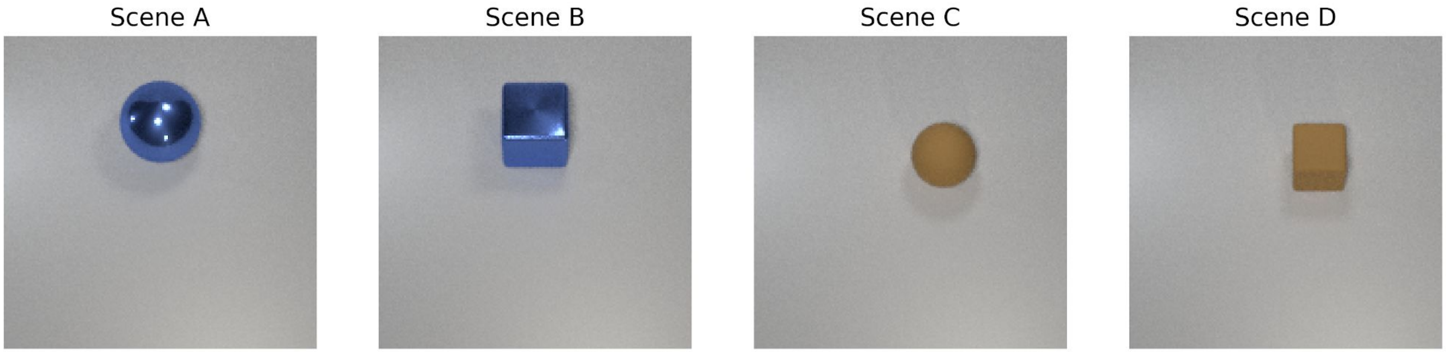
Bit   Bit

"Spot" : "Jane" : : "Fido" : "John"

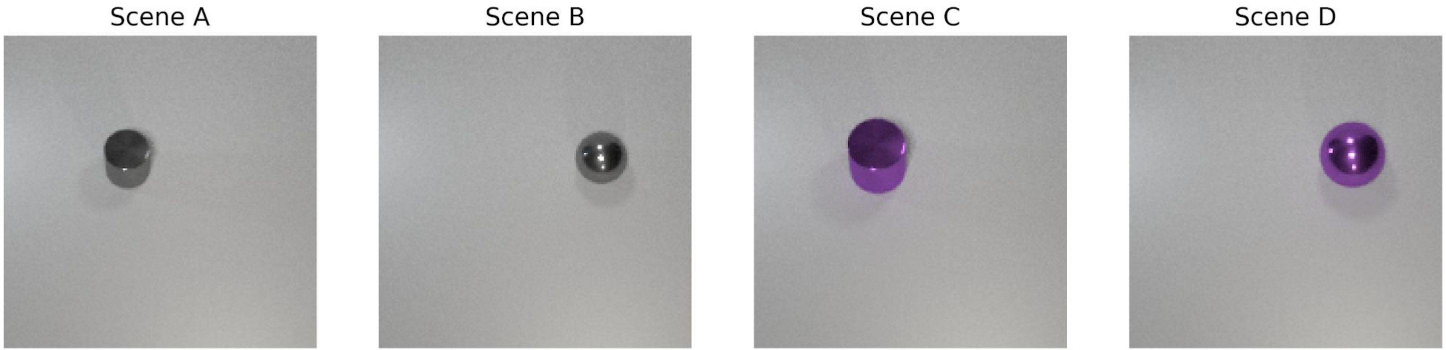
Simplifying things

`A : B :: C : D`

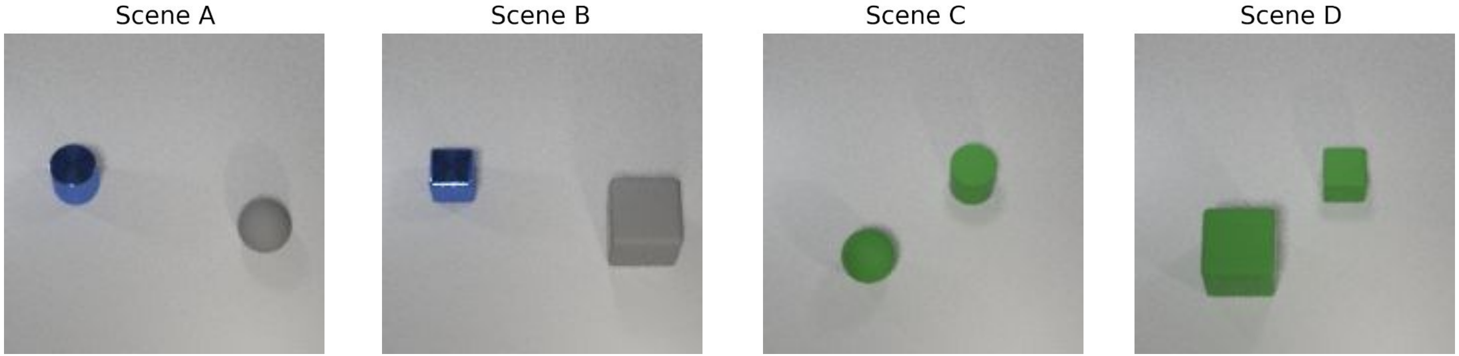
Visual Analogies



Visual Analogies



Visual Analogies



$$\mathtt{A} : \mathtt{B} \qquad \mathtt{C} : \mathtt{D}$$

$$\mathtt{A} \sim \mathbf{BLUE} \otimes \mathbf{METAL} \otimes \mathbf{SPHERE} := \mathbf{S_A}$$

$$\mathtt{B} \sim \mathbf{BLUE} \otimes \mathbf{METAL} \otimes \mathbf{CUBE} := \mathbf{S_B}$$

$$\mathtt{C} \sim \mathbf{BROWN} \otimes \mathbf{RUBBER} \otimes \mathbf{SPHERE} := \mathbf{S_C}$$

$$\mathtt{D} \sim \mathbf{BROWN} \otimes \mathbf{RUBBER} \otimes \mathbf{CUBE} := \mathbf{S_D}$$

Representing relations with bind/unbinding

$$\mathtt{A} : \mathtt{B} \sim \mathbf{S_A^{-1}} \otimes \mathbf{S_B} = \mathbf{SPHERE}^{-1} \otimes \mathbf{CUBE}$$

$$\mathtt{C} : \mathtt{D} \sim \mathbf{S_C^{-1}} \otimes \mathbf{S_D} = \mathbf{SPHERE}^{-1} \otimes \mathbf{CUBE}$$

$$\mathbf{S_A^{-1}} \otimes \mathbf{S_B} \otimes \mathbf{S_C} = \mathbf{S_D}$$

$$\mathtt{A} \sim \mathbf{BLUE} \otimes \mathbf{METAL} \otimes \mathbf{SPHERE} := \mathbf{S_A}$$

$$\mathtt{B} \sim \mathbf{BLUE} \otimes \mathbf{METAL} \otimes \mathbf{CUBE} := \mathbf{S_B}$$

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$$\mathtt{D} \sim \mathbf{BROWN} \otimes \mathbf{RUBBER} \otimes \mathbf{CUBE} := \mathbf{S_D}$$

$$\mathtt{A} : \mathtt{B} \qquad \mathtt{C} : \mathtt{D}$$

$$\mathtt{A} \sim \mathbf{COLOR} \otimes \mathbf{BLUE} \ + \ \mathbf{MATERIAL} \otimes \mathbf{METAL} \ + \ \mathbf{SHAPE} \otimes \mathbf{SPHERE} := \mathbf{S_A}$$

$$\mathtt{B} \sim \mathbf{COLOR} \otimes \mathbf{BLUE} \ + \ \mathbf{MATERIAL} \otimes \mathbf{METAL} \ + \ \mathbf{SHAPE} \otimes \mathbf{CUBE} := \mathbf{S_B}$$

$$\mathtt{C} \sim \mathbf{COLOR} \otimes \mathbf{BROWN} \ + \ \mathbf{MATERIAL} \otimes \mathbf{RUBBER} \ + \ \mathbf{SHAPE} \otimes \mathbf{SPHERE} := \mathbf{S_C}$$

$$\mathtt{D} \sim \mathbf{COLOR} \otimes \mathbf{BROWN} \ + \ \mathbf{MATERIAL} \otimes \mathbf{RUBBER} \ + \ \mathbf{SHAPE} \otimes \mathbf{CUBE} := \mathbf{S_D}$$

Representing relations with addition/subtraction

$$\mathtt{A} : \mathtt{B} \sim \mathbf{S_B} - \mathbf{S_A} = \mathbf{SHAPE} \otimes (\mathbf{CUBE} - \mathbf{SPHERE})$$

$$\mathtt{C} : \mathtt{D} \sim \mathbf{S_D} - \mathbf{S_C} = \mathbf{SHAPE} \otimes (\mathbf{CUBE} - \mathbf{SPHERE})$$

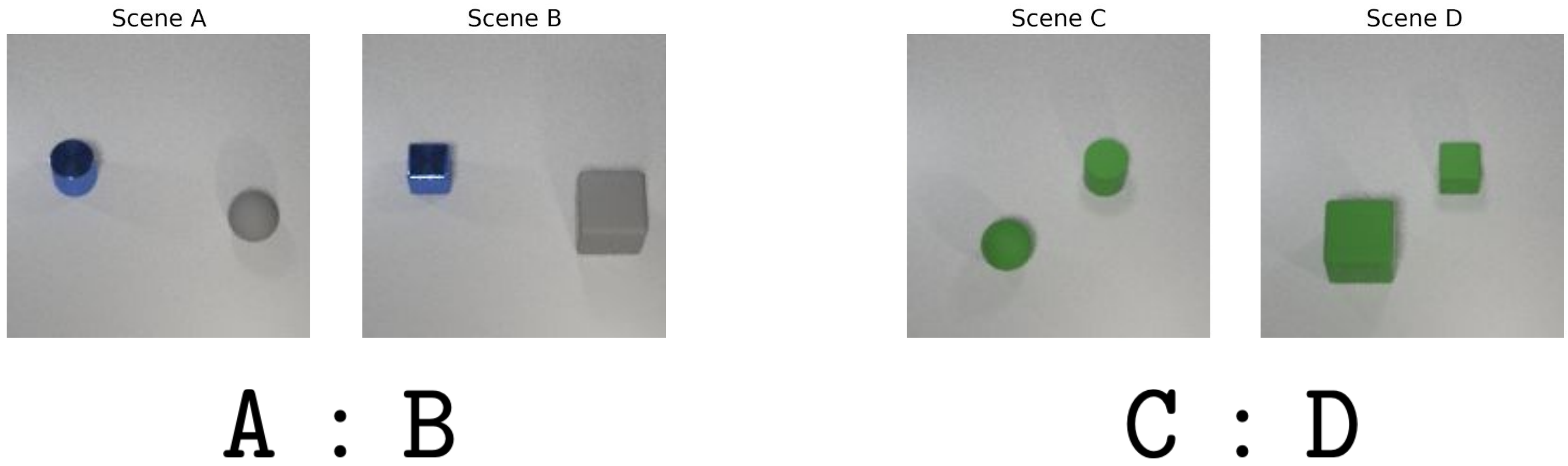
$$\mathbf{S_C} + \mathbf{S_B} - \mathbf{S_A} = \mathbf{S_D}$$

$$\mathtt{A} \sim \mathbf{COLOR} \otimes \mathbf{BLUE} + \mathbf{MATERIAL} \otimes \mathbf{METAL} + \mathbf{SHAPE} \otimes \mathbf{SPHERE} := \mathbf{S_A}$$

$$\mathtt{B} \sim \mathbf{COLOR} \otimes \mathbf{BLUE} + \mathbf{MATERIAL} \otimes \mathbf{METAL} + \mathbf{SHAPE} \otimes \mathbf{CUBE} := \mathbf{S_B}$$

$$\mathtt{C} \sim \mathbf{COLOR} \otimes \mathbf{BROWN} + \mathbf{MATERIAL} \otimes \mathbf{RUBBER} + \mathbf{SHAPE} \otimes \mathbf{SPHERE} := \mathbf{S_C}$$

$$\mathtt{D} \sim \mathbf{COLOR} \otimes \mathbf{BROWN} + \mathbf{MATERIAL} \otimes \mathbf{RUBBER} + \mathbf{SHAPE} \otimes \mathbf{CUBE} := \mathbf{S_D}$$



$$\mathtt{A} \sim \mathbf{SMALL} \otimes \mathbf{BLUE} \otimes \mathbf{METAL} \otimes \mathbf{CYLINDER} + \rho\Big(\mathbf{SMALL} \otimes \mathbf{GRAY} \otimes \mathbf{RUBBER} \otimes \mathbf{SPHERE}\Big) := \mathbf{S_A}$$

$$\mathtt{B} \sim \mathbf{SMALL} \otimes \mathbf{BLUE} \otimes \mathbf{METAL} \otimes \mathbf{CUBE} + \rho\Big(\mathbf{LARGE} \otimes \mathbf{GRAY} \otimes \mathbf{RUBBER} \otimes \mathbf{CUBE}\Big) := \mathbf{S_B}$$

$$\mathtt{C} \sim \mathbf{SMALL} \otimes \mathbf{GREEN} \otimes \mathbf{RUBBER} \otimes \mathbf{CYLINDER} + \rho\Big(\mathbf{SMALL} \otimes \mathbf{GREEN} \otimes \mathbf{RUBBER} \otimes \mathbf{SPHERE}\Big) := \mathbf{S_C}$$

$$\mathtt{D} \sim \mathbf{SMALL} \otimes \mathbf{GREEN} \otimes \mathbf{RUBBER} \otimes \mathbf{CUBE} + \rho\Big(\mathbf{LARGE} \otimes \mathbf{GREEN} \otimes \mathbf{RUBBER} \otimes \mathbf{CUBE}\Big) := \mathbf{S_D}$$

Representing relations with bind/unbinding

$$\mathtt{A} : \mathtt{B} \sim \mathbf{S_A^{-1}} \otimes \mathbf{S_B} = \mathbf{CYLINDER}^{-1} \otimes \mathbf{CUBE} + \rho\left(\mathbf{SMALL}^{-1} \otimes \mathbf{LARGE} \otimes \mathbf{SPHERE}^{-1} \otimes \mathbf{CUBE}\right) + \boldsymbol{\eta_1}$$

$$\mathtt{C} : \mathtt{D} \sim \mathbf{S_C^{-1}} \otimes \mathbf{S_D} = \mathbf{CYLINDER}^{-1} \otimes \mathbf{CUBE} + \rho\left(\mathbf{SMALL}^{-1} \otimes \mathbf{LARGE} \otimes \mathbf{SPHERE}^{-1} \otimes \mathbf{CUBE}\right) + \boldsymbol{\eta_2}$$

$$\mathbf{S_A^{-1}} \otimes \mathbf{S_B} \otimes \mathbf{S_C} = \mathbf{S_D} + \boldsymbol{\eta_3}$$

$\boldsymbol{\eta_*}$ represents *noise*

$$\mathtt{A} \sim \mathbf{SMALL} \otimes \mathbf{BLUE} \otimes \mathbf{METAL} \otimes \mathbf{CYLINDER} + \rho\left(\mathbf{SMALL} \otimes \mathbf{GRAY} \otimes \mathbf{RUBBER} \otimes \mathbf{SPHERE}\right) := \mathbf{S_A}$$

$$\mathtt{B} \sim \mathbf{SMALL} \otimes \mathbf{BLUE} \otimes \mathbf{METAL} \otimes \mathbf{CUBE} + \rho\left(\mathbf{LARGE} \otimes \mathbf{GRAY} \otimes \mathbf{RUBBER} \otimes \mathbf{CUBE}\right) := \mathbf{S_B}$$

$$\mathtt{C} \sim \mathbf{SMALL} \otimes \mathbf{GREEN} \otimes \mathbf{RUBBER} \otimes \mathbf{CYLINDER} + \rho\left(\mathbf{SMALL} \otimes \mathbf{GREEN} \otimes \mathbf{RUBBER} \otimes \mathbf{SPHERE}\right) := \mathbf{S_C}$$

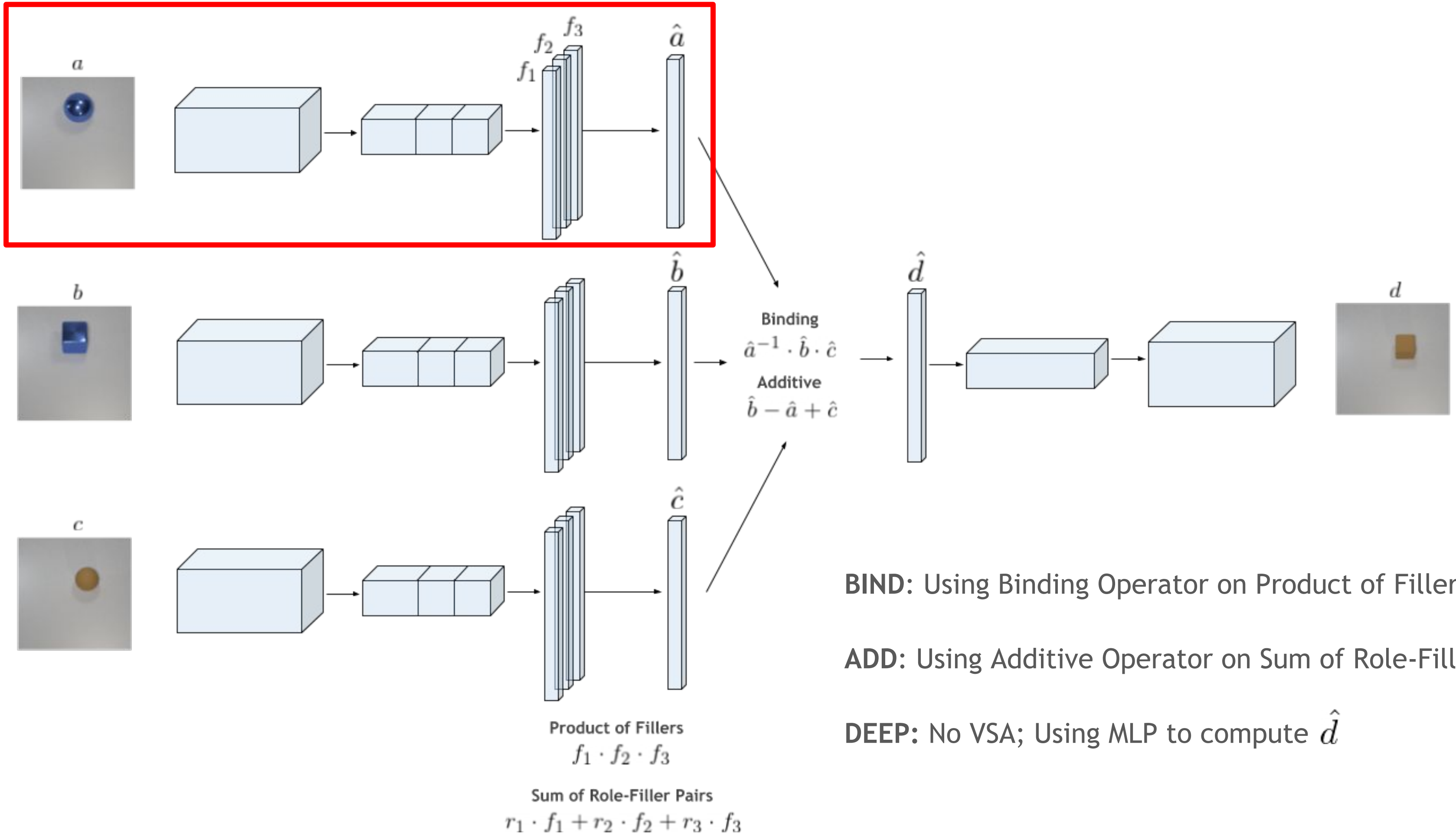
$$\mathtt{D} \sim \mathbf{SMALL} \otimes \mathbf{GREEN} \otimes \mathbf{RUBBER} \otimes \mathbf{CUBE} + \rho\left(\mathbf{LARGE} \otimes \mathbf{GREEN} \otimes \mathbf{RUBBER} \otimes \mathbf{CUBE}\right) := \mathbf{S_D}$$

VSA encodings make analogical reasoning transparent and elegant

- Relations are explicitly depicted in terms of atomic factors
- Simple binding/additive operations are used to reason about analogies

...and they can be learned with neural networks

Overview of Model Architecture

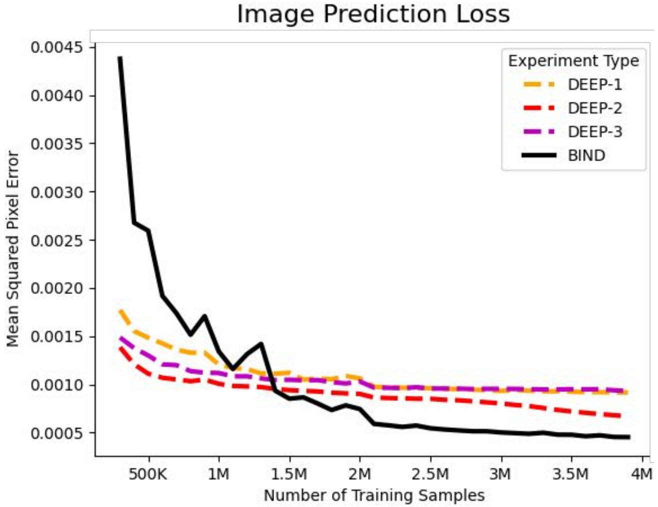


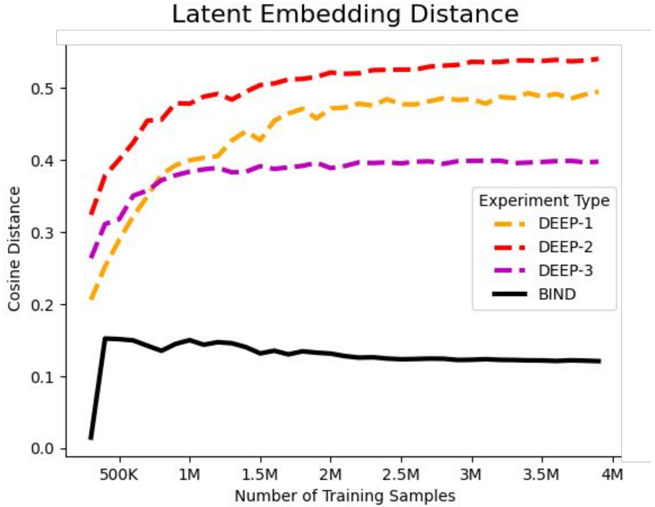
BIND: Using Binding Operator on Product of Fillers

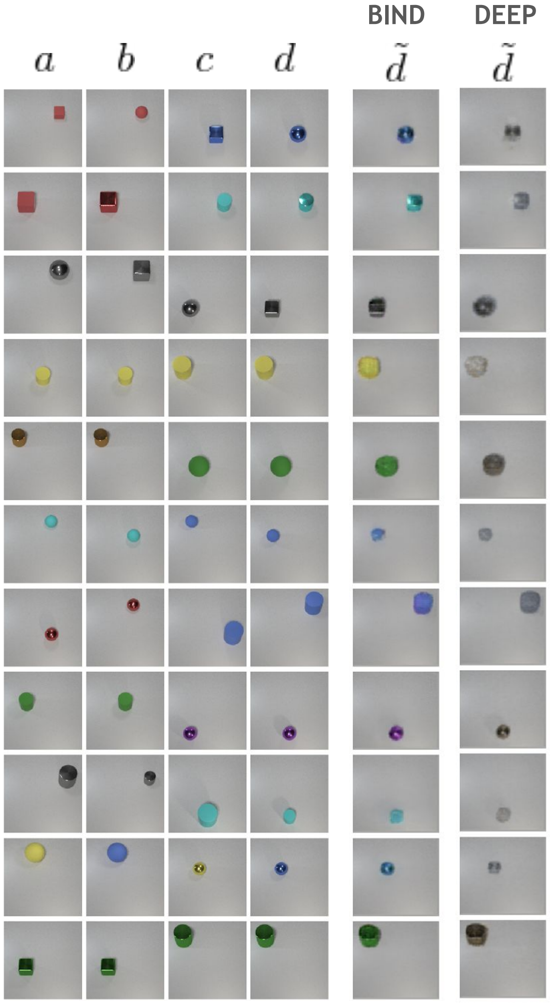
ADD: Using Additive Operator on Sum of Role-Filler Pairs

DEEP: No VSA; Using MLP to compute $\hat{d}$

End-to-End Experiments





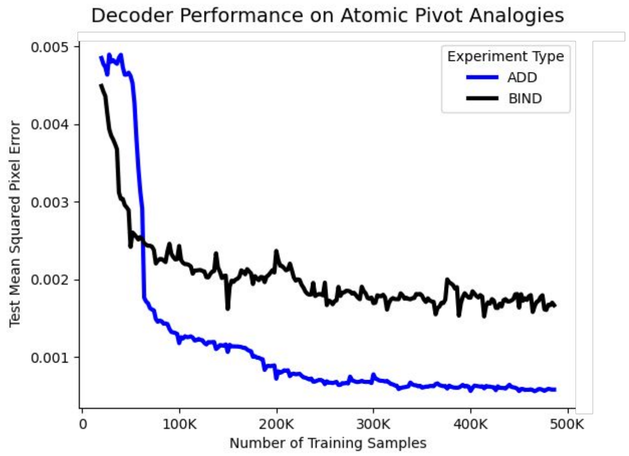


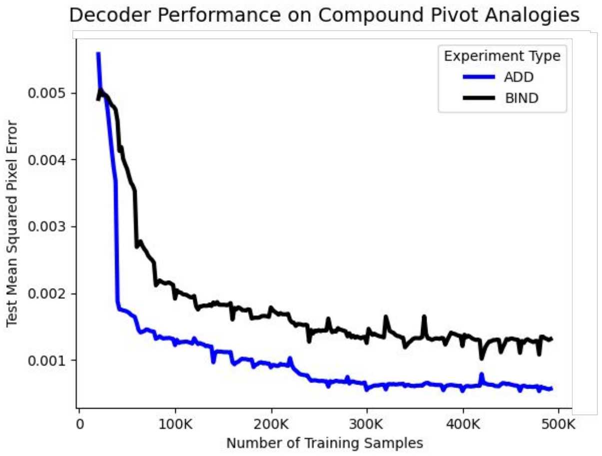
BIND: $1 - sim(\hat{a}^{-1} \cdot \hat{b} \cdot \hat{c}, \hat{d})$

DEEP: $1 - sim(MLP(\hat{a}, \hat{b}, \hat{c}), \hat{d})$

The Difficulty of Decoding

How do different VSA representations affect decoder performance?





Filler variables are easily accessible using sum of Role-Filler Pairs representation

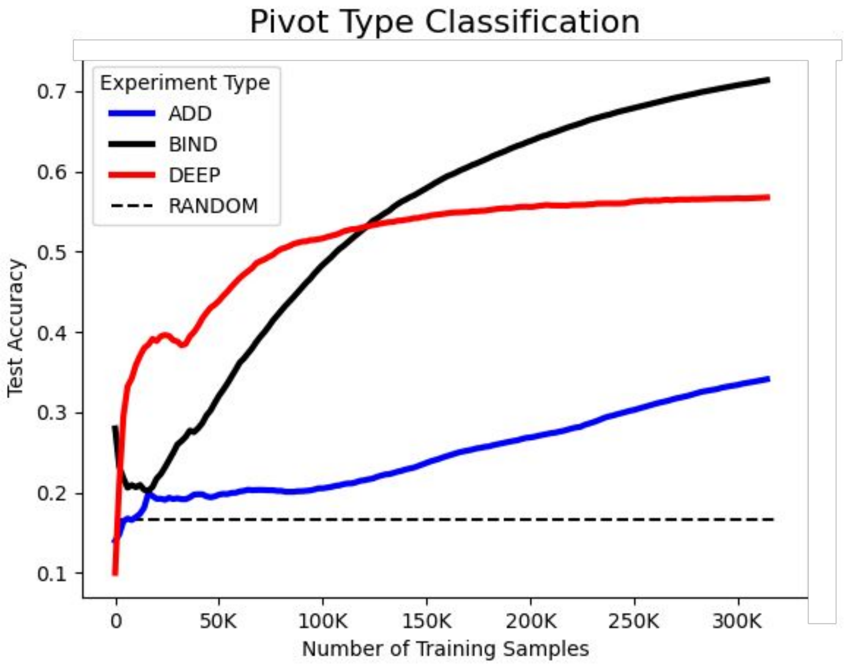
$$r_1^{-1} \cdot (r_1 \cdot f_1 + r_2 \cdot f_2 + r_3 \cdot f_3) \approx f_1$$

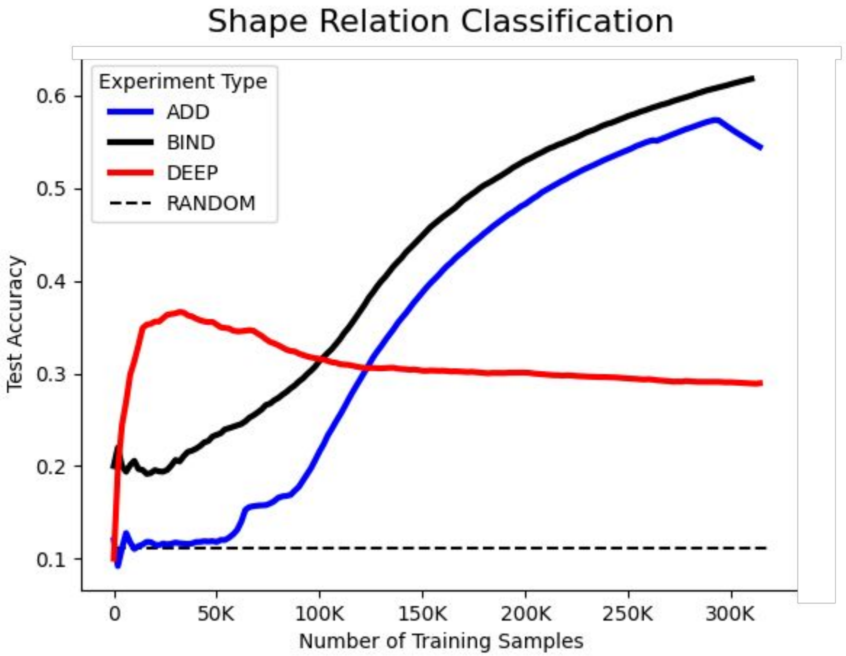
Difficult for deep networks to factorize Product of Fillers representation, but better solutions exist [1][2]

[1] Frady, E., Kent, S., Olshausen, B., & Sommer, F. (2020). Resonator Networks, 1. *Neural Computation, to appear (preprint: arXiv:2007.03748)*
[2] Kent, S., Frady, E., Sommer, F., & Olshausen, B. (2020). Resonator Networks, 2: *Neural Computation, to appear (preprint: arXiv:1906.11684)*

Encoder for Relation Classification

Do structured representations efficiently uncover the relation defining the analogy?





Conclusion

- Representing objects with structured representations (e.g. VSAs) makes modeling and reasoning about analogies simple and transparent
- Reasoning is downstream from the encoding problem.
- Deep learning can be used to learn these structured representations and support analogical reasoning.
    - Work to be done on 1) self-supervised (and decoding-free) training strategies and 2) principles of design for deep learning encoders.

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Extra slides

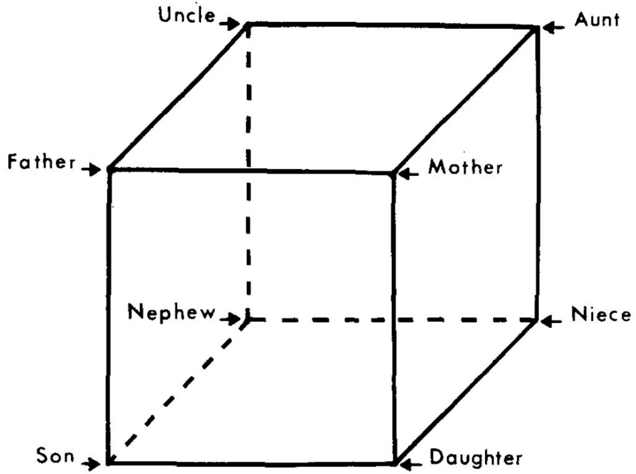


FIG. 2. Three-dimensional representation of relations among eight kinship terms.

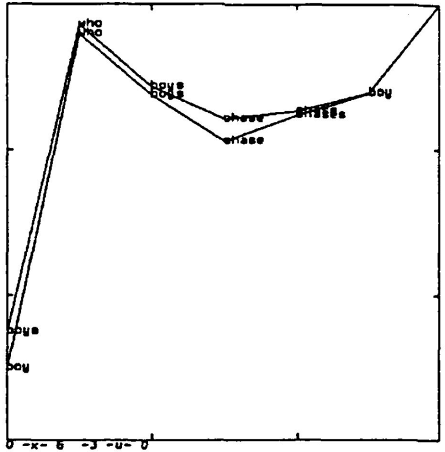


Figure 8

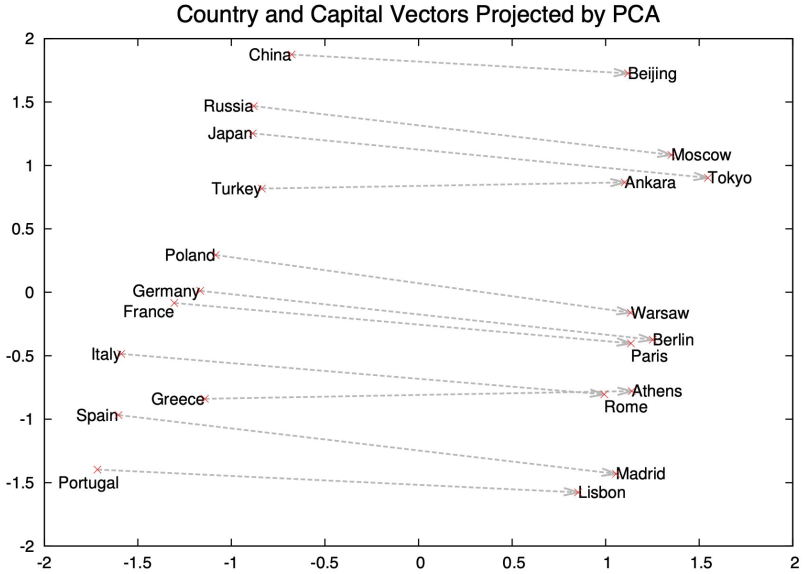
Trajectories through state space for sentences (8c) and (8d).

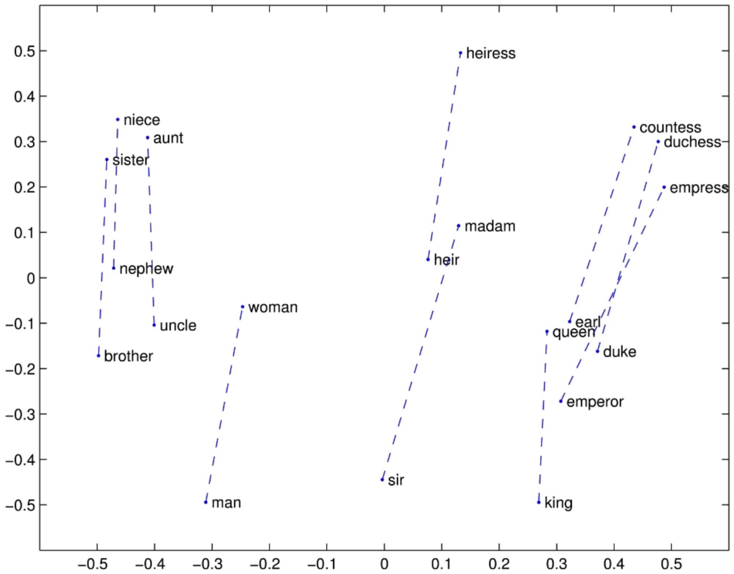
(8c) boy who boys chase chases boy .

(8d) boys who boys chase chase boy .

[1] Rumelhart, D. E., & Abrahamson, A. A. (1973). A model for analogical reasoning. *Cognitive Psychology*, *5*(1), 1-28.
[2] Elman, J. L. (1989). Representation and structure in connectionist models. *UCSD La Jolla Center for Research in Language*.

Analogies on encodings of words





[1] Mikolov, T., Chen, K., Corrado, G., & Dean, J. (2013). *arXiv:1301.3781*.
[2] Mikolov, T., Yih, W. T., & Zweig, G. (2013). *NAACL-HLT*
[3] Mikolov, T., Sutskever, I., Chen, K., Corrado, G. S., & Dean, J. (2013). *NeurIPS*.
[4] Pennington, J., Socher, R., & Manning, C. D. (2014). *EMNLP*.

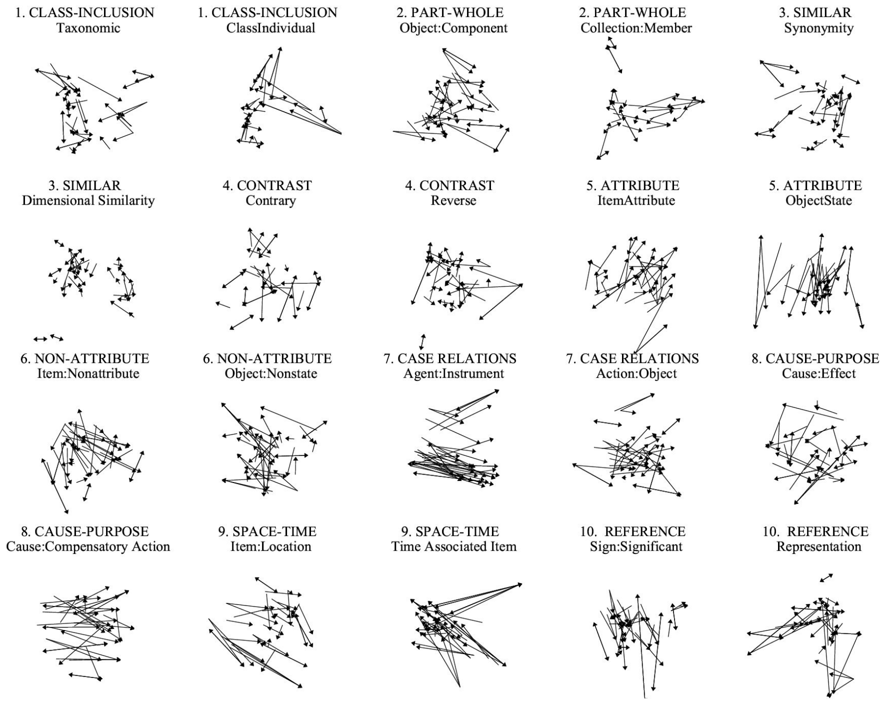
"Somewhat surprisingly, many of these patterns can be represented as linear translations." [1]

"Since vector spaces are inherently linear structures, the most natural way to do this is with vector differences" [2]

[1] Mikolov, T., Sutskever, I., Chen, K., Corrado, G. S., & Dean, J. (2013). *NeurIPS*.
[2] Pennington, J., Socher, R., & Manning, C. D. (2014). *EMNLP*.

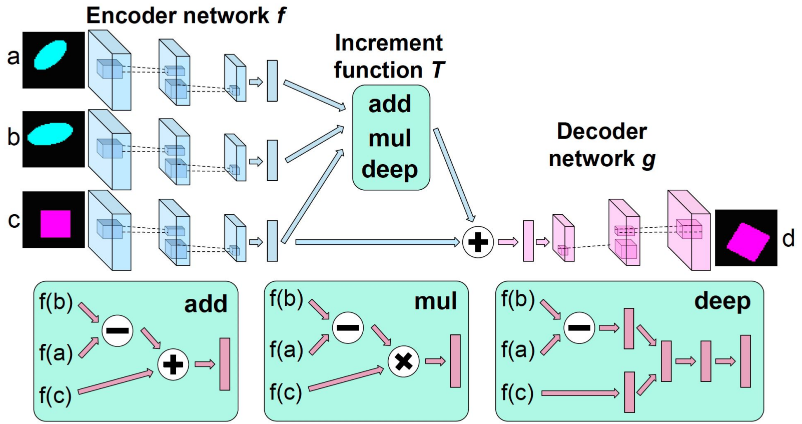
Why should relations be captured with sum/difference?

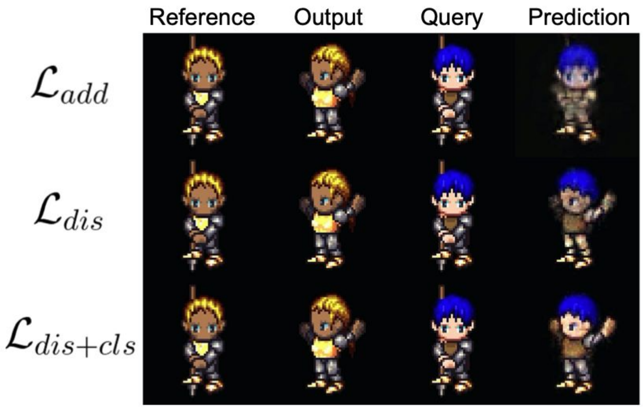
	Add	Multiply	Only–b	Ignore–a	Add–opposite	Vanilla	Reversed (Add)	Reversed (Only–b)
Common capitals	.90	.92	.13	.62	.00	.05	.53	.04
All capitals	.77	.80	.17	.37	.00	.01	.57	.08
US cities	.69	.69	.25	.30	.01	.00	.17	.08
Currencies	.13	.15	.00	.08	.00	.03	.12	.00
Nationalities	.88	.89	.29	.69	.00	.21	.97	.54
Gender	.78	.79	.31	.37	.07	.04	.82	.22
Singular to plural	.80	.80	.70	.49	.45	.00	.71	.60
Base to gerund	.66	.67	.52	.37	.24	.00	.71	.64
Gerund to past	.57	.63	.17	.25	.06	.00	.46	.15
Base to third person	.60	.67	.20	.32	.07	.00	.69	.40
Adj. to adverb	.33	.34	.22	.14	.05	.00	.23	.16
Adj. to comparative	.86	.86	.36	.50	.00	.00	.59	.17
Adj. to superlative	.59	.69	.03	.19	.00	.00	.43	.15
Adj. un– prefixation	.38	.39	.17	.12	.01	.00	.36	.24



[1] Linzen, T. (2016). Issues in evaluating semantic spaces using word analogies. *arXiv preprint arXiv:1606.07736*.
[2] Chen, D., Peterson, J. C., & Griffiths, T. L. (2017). Evaluating vector-space models of analogy. *arXiv preprint arXiv:1705.04416*.

Visual analogies





[1] Reed, S. E., Zhang, Y., Zhang, Y., & Lee, H. (2015). Deep visual analogy-making. *NeurIPS*.

References for extra slides

Chen, D., Peterson, J. C., & Griffiths, T. L. (2017). Evaluating vector-space models of analogy. *arXiv preprint arXiv:1705.04416*.

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