Analogical Reasoning with VSAs/HD computing
What is an analogy?

“Analogies are partial similarities between different situations that support further inference”
--Dedre Gentner

“...the very blue that fills the whole sky of cognition--analogy is everything, or very nearly so, in my view.”
--Douglas Hofstadter
What is an analogy?

“Spot bit Jane”

“Fido bit John”

“Jane bit Fido”

...an analogy captures *correspondences* between each of these instances
What is an analogy?

Propositions

“Spot bit Jane”

“Fido bit John”

“Jane bit Fido”

bite(Jane, Fido)

predicate(agent, object)
What is an analogy?

Four (give or take) tasks of analogical reasoning.

1. Retrieval
   ○ Accessing analogous instances from memory--unconscious and fast.

2. Judgement
   ○ Estimating the similarity of different instances based on structural and semantic properties.
     Conscious and slow.

3. Mapping
   ○ Finding structural correspondences between instances

4. Inference
   ○ Applying structure from one instance to a generate or explain a novel instance

Hummel & Holyoak (1997). Psychological Review
Propositional analogies with HRRs

- Encoding depends on the analogical reasoning task
- Example (from Tony Plate):
  - HRRs encodings that reflect human judgements of similarity

Plate, T. (2000). Analogy retrieval and processing with distributed vector representations
Propositional analogies with HRRs

“Spot bit Jane, causing Jane to flee from Spot”

\[ P_{\text{bite}} = \text{bite} + \text{spot} + \text{jane} + \text{bite}_{\text{agt}} \otimes \text{spot} + \text{bite}_{\text{obj}} \otimes \text{jane} \]

\[ P_{\text{flee}} = \text{flee} + \text{spot} + \text{jane} + \text{flee}_{\text{agt}} \otimes \text{jane} + \text{flee}_{\text{from}} \otimes \text{spot} \]

\[ P = \text{cause} + P_{\text{bite}} + P_{\text{flee}} + \text{cause}_{\text{antc}} \otimes P_{\text{bite}} + \text{cause}_{\text{cnsq}} \otimes P_{\text{flee}} \]
Propositional analogies with HRRs

“Spot bit Jane, causing Jane to flee from Spot”
Propositional analogies with HRRs

Different types of “similarity”

- **LS** (literal similarity): ‘Fido bit John, causing John to flee from Fido’. (Has both structural and superficial similarity to the probe $P$.)
- **SF** (surface features): ‘John fled from Fido, causing Fido to bite John’. (Has superficial but not structural similarity, but types of corresponding objects are switched.)
- **AN** (analogy): ‘Mort bit Felix, causing Felix to flee from Mort’. (Has structural but not superficial similarity.)
- **FOR** (first-order relations only): ‘Mort fled from Felix, causing Felix to bite Mort’. (Has neither structural nor superficial similarity, other than shared predicates.)
Propositional analogies with HRRs

Table 2

<table>
<thead>
<tr>
<th>Episodes in long-term memory</th>
<th>Commonalities with probe</th>
<th>Similarity scores</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>HRR</td>
</tr>
<tr>
<td>( E_{LS} ) Fido bit John, causing John to flee from Fido</td>
<td>✓ ✓ ✓</td>
<td>0.71</td>
</tr>
<tr>
<td>( E_{SF} ) John fled from Fido, causing Fido to bite John</td>
<td>✓ ✓ x</td>
<td>0.47</td>
</tr>
<tr>
<td>( E_{CM} ) Fred bit Rover, causing Rover to flee from Fred</td>
<td>✓ ✓ ✓</td>
<td>0.47</td>
</tr>
<tr>
<td>( E_{AN} ) Mort bit Felix, causing Felix to flee from Mort</td>
<td>x ✓ ✓</td>
<td>0.42</td>
</tr>
<tr>
<td>( E_{FOR} ) Mort fled from Felix, causing Felix to bite Mort</td>
<td>x ✓ x</td>
<td>0.30</td>
</tr>
</tbody>
</table>

\( P \), Spot bit Jane, causing Jane to flee from Spot.
Issues with what we’ve shown so far

- Complicated
- Hand-designed
- How do atomic concepts (vectors) get their meaning?
- Evaluated similarity without further decomposition
Simplifying things

A : B :: C : D
Simplifying things

Bit
“Spot” : “Jane” :: “Fido” : “John”
Simplifying things

\[ A : B :: C : D \]
Visual Analogies
Visual Analogies
Visual Analogies
Scene A  Scene B  Scene C  Scene D

A : B  C : D

A \sim \text{BLUE} \otimes \text{METAL} \otimes \text{SPHERE} := S_A

B \sim \text{BLUE} \otimes \text{METAL} \otimes \text{CUBE} := S_B

C \sim \text{BROWN} \otimes \text{RUBBER} \otimes \text{SPHERE} := S_C

D \sim \text{BROWN} \otimes \text{RUBBER} \otimes \text{CUBE} := S_D
Representing relations with bind/unbinding

\[ A : B \sim S_A^{-1} \otimes S_B = \text{SPHERE}^{-1} \otimes \text{CUBE} \]
\[ C : D \sim S_C^{-1} \otimes S_D = \text{SPHERE}^{-1} \otimes \text{CUBE} \]

\[ S_A^{-1} \otimes S_B \otimes S_C = S_D \]

\[ A \sim \text{BLUE} \otimes \text{METAL} \otimes \text{SPHERE} := S_A \]
\[ B \sim \text{BLUE} \otimes \text{METAL} \otimes \text{CUBE} := S_B \]
\[ C \sim \text{BROWN} \otimes \text{RUBBER} \otimes \text{SPHERE} := S_C \]
\[ D \sim \text{BROWN} \otimes \text{RUBBER} \otimes \text{CUBE} := S_D \]
A : B

A \sim \text{COLOR} \otimes \text{BLUE} + \text{MATERIAL} \otimes \text{METAL} + \text{SHAPE} \otimes \text{SPHERE} := S_A

B \sim \text{COLOR} \otimes \text{BLUE} + \text{MATERIAL} \otimes \text{METAL} + \text{SHAPE} \otimes \text{CUBE} := S_B

C \sim \text{COLOR} \otimes \text{BROWN} + \text{MATERIAL} \otimes \text{RUBBER} + \text{SHAPE} \otimes \text{SPHERE} := S_C

D \sim \text{COLOR} \otimes \text{BROWN} + \text{MATERIAL} \otimes \text{RUBBER} + \text{SHAPE} \otimes \text{CUBE} := S_D
Representing relations with addition/subtraction

\[ A : B \sim S_B - S_A = SHAPE \otimes (CUBE - SPHERE) \]

\[ C : D \sim S_D - S_C = SHAPE \otimes (CUBE - SPHERE) \]

\[ S_C + S_B - S_A = S_D \]

\[ A \sim COLOR \otimes BLUE + MATERIAL \otimes METAL + SHAPE \otimes SPHERE := S_A \]

\[ B \sim COLOR \otimes BLUE + MATERIAL \otimes METAL + SHAPE \otimes CUBE := S_B \]

\[ C \sim COLOR \otimes BROWN + MATERIAL \otimes RUBBER + SHAPE \otimes SPHERE := S_C \]

\[ D \sim COLOR \otimes BROWN + MATERIAL \otimes RUBBER + SHAPE \otimes CUBE := S_D \]
$A \sim \text{SMALL } \otimes \text{BLUE } \otimes \text{METAL } \otimes \text{CYLINDER} + \rho\left(\text{SMALL } \otimes \text{GRAY } \otimes \text{RUBBER } \otimes \text{SPHERE}\right) := S_A$

$B \sim \text{SMALL } \otimes \text{BLUE } \otimes \text{METAL } \otimes \text{CUBE} + \rho\left(\text{LARGE } \otimes \text{GRAY } \otimes \text{RUBBER } \otimes \text{CUBE}\right) := S_B$

$C \sim \text{SMALL } \otimes \text{GREEN } \otimes \text{RUBBER } \otimes \text{CYLINDER} + \rho\left(\text{SMALL } \otimes \text{GREEN } \otimes \text{RUBBER } \otimes \text{SPHERE}\right) := S_C$

$D \sim \text{SMALL } \otimes \text{GREEN } \otimes \text{RUBBER } \otimes \text{CUBE} + \rho\left(\text{LARGE } \otimes \text{GREEN } \otimes \text{RUBBER } \otimes \text{CUBE}\right) := S_D$
Representing relations with bind/unbinding

\[ A : B \sim S_A^{-1} \otimes S_B = \text{CYLINDER}^{-1} \otimes \text{CUBE} + \rho(\text{SMALL}^{-1} \otimes \text{LARGE} \otimes \text{SPHERE}^{-1} \otimes \text{CUBE}) + \eta_1 \]

\[ C : D \sim S_C^{-1} \otimes S_D = \text{CYLINDER}^{-1} \otimes \text{CUBE} + \rho(\text{SMALL}^{-1} \otimes \text{LARGE} \otimes \text{SPHERE}^{-1} \otimes \text{CUBE}) + \eta_2 \]

\[ S_A^{-1} \otimes S_B \otimes S_C = S_D + \eta_3 \]

*\( \eta_\star \) represents noise

\[ A \sim \text{SMALL} \otimes \text{BLUE} \otimes \text{METAL} \otimes \text{CYLINDER} + \rho(\text{SMALL} \otimes \text{GRAY} \otimes \text{RUBBER} \otimes \text{SPHERE}) := S_A \]

\[ B \sim \text{SMALL} \otimes \text{BLUE} \otimes \text{METAL} \otimes \text{CUBE} + \rho(\text{LARGE} \otimes \text{GRAY} \otimes \text{RUBBER} \otimes \text{CUBE}) := S_B \]

\[ C \sim \text{SMALL} \otimes \text{GREEN} \otimes \text{RUBBER} \otimes \text{CYLINDER} + \rho(\text{SMALL} \otimes \text{GREEN} \otimes \text{RUBBER} \otimes \text{SPHERE}) := S_C \]

\[ D \sim \text{SMALL} \otimes \text{GREEN} \otimes \text{RUBBER} \otimes \text{CUBE} + \rho(\text{LARGE} \otimes \text{GREEN} \otimes \text{RUBBER} \otimes \text{CUBE}) := S_D \]
VSA encodings make analogical reasoning transparent and elegant

- Relations are explicitly depicted in terms of atomic factors
- Simple binding/additive operations are used to reason about analogies

...and they can be learned with neural networks
Overview of Model Architecture

**BIND**: Using Binding Operator on Product of Fillers

**ADD**: Using Additive Operator on Sum of Role-Filler Pairs

**DEEP**: No VSA; Using MLP to compute $\hat{d}$
End-to-End Experiments

Image Prediction Loss

Latent Embedding Distance

BIND: $1 - \text{sim}(\hat{a}^{-1} \cdot \hat{b} \cdot \hat{c}, \hat{d})$

DEEP: $1 - \text{sim}(\text{MLP}(\hat{a}, \hat{b}, \hat{c}), \hat{d})$
The Difficulty of Decoding

How do different VSA representations affect decoder performance?

Filler variables are easily accessible using sum of Role-Filler Pairs representation

\[ r_1^{-1} \cdot (r_1 \cdot f_1 + r_2 \cdot f_2 + r_3 \cdot f_3) \approx f_1 \]

Difficult for deep networks to factorize Product of Fillers representation, but better solutions exist [1][2]

Encoder for Relation Classification

Do structured representations efficiently uncover the relation defining the analogy?

**Pivot Type Classification**

**Shape Relation Classification**
Conclusion

- Representing objects with structured representations (e.g. VSAs) makes modeling and reasoning about analogies simple and transparent.
- Reasoning is downstream from the encoding problem.
- Deep learning can be used to learn these structured representations and support analogical reasoning.
  - Work to be done on 1) self-supervised (and decoding-free) training strategies and 2) principles of design for deep learning encoders.
References


Extra slides
Fig. 2. Three-dimensional representation of relations among eight kinship terms.

(8c) boy who boys chase chases boy.
(8d) boys who boys chase chase boy.

Analogy on encodings of words

![Graph showing country and capital vectors projected by PCA]

“Somewhat surprisingly, many of these patterns can be represented as linear translations.” [1]

“Since vector spaces are inherently linear structures, the most natural way to do this is with vector differences” [2]
Why should relations be captured with sum/difference?


Visual analogies

References for extra slides


