VS 265: Neural Computation **Fall 2016**

Dynamics

Redwood Center for Theoretical Neuroscience

1 Differential Equations

- *Dynamics* is essentially the study of how things change over time. This is important for understanding the brain, because we are constantly being inundated with time-varying signals. How neurons respond to these signals over time is the essence of what neural coding is all about.
- *Differential* equations provide a mathematical description for how things change over time. The central element of a differential equation is the *time-derivative* of a variable, $\frac{dx}{dt}$. This essentially tells us the rate of change of the variable x, where $x(t)$ may be voltage, position of a particle, etc.
- The derivative can have different orders: $\frac{dx}{dt}$, $\frac{d^2x}{dt^2} = \frac{d}{dt}(\frac{dx}{dt})$, $\frac{d^3x}{dt^3} = \frac{d}{dt}(\frac{d^2x}{dt^2})$, etc. For example, if x is position, then $\frac{dx}{dt}$ will be its velocity and $\frac{d^2x}{dt^2}$ its acceleration.
- A *first-order* differential equation would have terms of the form x and $\frac{dx}{dt}$. A *second-order* differential equation would have terms of the form x, $\frac{dx}{dt}$, and $\frac{d^2x}{dt^2}$. Or in general, an nth-order differential equation would have terms of the form x, $\frac{dx}{dt}$, ..., $\frac{d^nx}{dt^n}$
- The simplest kind of differential equation is a *linear differential equation*, which simply contains derivatives of different orders, each multiplied by a constant:

$$a_0x + a_1\frac{dx}{dt} + a_2\frac{d^2x}{dt^2} + ... + a_n\frac{d^nx}{dt^n} = 0$$

- Oftentimes, we denote a time-derivative using the 'dot' notation, $\dot{x} \equiv \frac{dx}{dt}$. Thus, in this notation, a second-order linear differential equation would have the form

$$a_0x + a_1\dot{x} + a_2\ddot{x} = 0$$

2 Exponential Decay

- The very simplest differential equation would be a first-order, linear differential equation:

$$a_0x + a_1\dot{x} = 0$$

- When $a_0 = a_1 = 1$, we have the simple relation $\dot{x} = -x$. What this tells us is that the rate of change of x depends on the value of x. If x is currently a large positive value, then x will decrease quickly. If x is currently a negative value, then x will increase. Is there a mathematical equation that will tell us explicitly how x changes as a function of time? Yes, it turns out that the general solution to the above equation is of the form:

$$x(t) = ke^{-t/\tau}$$

where the constant k is determined by the initial condition, or the initial state of x at $t = 0$, and $\tau = a_1/a_0$. For example, if $x(0) = 1$ and $a_0 = a_1 = 1$, then we have $x(t) = e^{-t}$. Thus, a first-order linear differential equation describes the process of *exponential decay.*

- The *rate* of decay is determined by the 'time constant,' τ. If τ is large, then this means that x decays slowly. If τ is very small, then x decays quickly. Basically, the way to think of it is that when an amount of time τ has gone by, the value of x will have been reduced by a factor of $1/e$ (the number e is about 2.7).

3 Leaky integrator

So far, we have examined the case where the right-hand side of the differential equation is zero. What if there is a time-varying function on the right hand side? i.e.

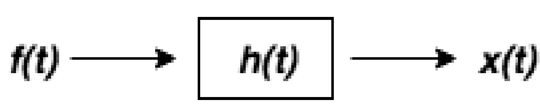
$$a_0 x + a_1 \dot{x} = f(t)$$

How does $x(t)$ vary as a function of $f(t)$?

It turns out that the solution for $x(t)$ is now

$$x(t) = \int_{-\infty}^{t} f(T)h(t-T)dT := (f * h)(t)$$

where the function $h(t)$ is the solution obtained when the right-hand side is zero. Often the convolution operator ($*$) is used to denote this expression more simply as seen above. This equation tells us that $x(t)$ is a linearly weighted sum of the present and past values of $f(t)$. The weights are given by the function $h(t)$, which as we have seen above is an exponentially decaying function. So the more recent values of $f(t)$ will be weighted more heavily than the present values, and values far in the past will be entirely forgotten. Such a system is called a *leaky integrator*, because the past leaks away from the summation of present and past values of $f(t)$. We can think of $f(t)$ as the input to the system and $x(t)$ as the output of the system



where $h(t)$ characterizes, via its time-constant, how far in the past that values of $f(t)$ will affect the current value of $x(t)$. Thus, $h(t)$ acts as a filter on the function $f(t)$, smoothing over its details. If τ is large, then smoothing will be severe. If τ is small, little smoothing will occur.

3.1 Adding an equilibrium state

If we have the following differential equation:

$$b_0 \dot{x} + x = c + f(t)$$

where b_0 and c are constants then the solution is:

$$x(t) = c + \frac{1}{b_0} \int_{-\infty}^{t} f(T)h(t-T)dT$$

where the function $h(t)$ is the solution obtained when the right-hand side is zero.