Sparse coding via a locally competitive algorithm (LCA)

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Recall that the sparse coding image model is of the form

$$I(x,y) = \sum_i a_i\, \phi_i(x,y) + \epsilon(x,y) \tag{1}$$

where $I(x,y)$ denotes the pixel intensities within an image patch (x and y are spatial coordinates), and the $\phi_i(x,y)$ are a set of basis functions or 'features' for describing the image. The a_i correspond to neural activity (the subscript i denotes the index of the neuron). The goal is to find a set of features that allow images to be described using as few non-zero a_i as possible.

The energy function governing this system is as follows:

$$E = \frac{1}{2} \sum_{x,y} \left[I(x,y) - \sum_i a_i\, \phi_i(x,y) \right]^2 + \lambda \sum_i C(a_i) \tag{2}$$

The neural activities a_i for a given image $I(x,y)$ are computed by finding a minimum of this energy function. In the original sparse coding algorithm (Olshausen & Field 1996), they were computed by direct gradient descent. However this it not very efficient in terms of a numerical (algorithmic) implementation. In addition, we would like to have a *physical* implementation that corresponds more closely to what a neural population could do.

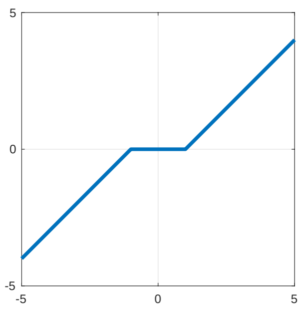
In Rozell et al. (2008), we derived a dynamical system for efficiently computing the a_i which descends the energy, but not according to steepest descent per se. This system is described by the following equations:

$$\begin{aligned} \tau\, \dot{u}_i + u_i &= b_i - \sum_{j \neq i} G_{ij}\, a_j & (3) \\ a_i &= g(u_i) & (4) \end{aligned}$$

where $b_i = \sum_{x,y} \phi_i(x,y)\, I(x,y)$ and $G_{ij} = \sum_{x,y} \phi_i(x,y)\, \phi_j(x,y)$. The nonlinearity $g()$ is determined by the form of the cost function $C()$. In the case where C is the L1 norm over the coefficients, i.e., $C(a_i) = |a_i|$, then g is a 'soft-thresholding' function of the form

$$g(u_i) = \begin{cases} u_i - \lambda & u_i > \lambda \\ 0 & -\lambda < u_i < \lambda \\ u_i + \lambda & u_i < -\lambda \end{cases} \tag{5}$$

This is plotted below for $\lambda = 1$. It is also called a "shrinkage" function because it shrinks the values below threshold to zero, and those above threshold by λ.



We called this method a "locally competitive algorithm" (LCA) because it computes the coefficients through a local competition (lateral inhibition) and thresholding. The LCA has a simple neural implementation as follows: each neuron is a leaky integrator that is driven by a feedforward term - the b_i, which is the inner product between the neuron's receptive field $\phi_i(x, y)$ and image $I(x, y)$ - and inhibited by other neurons in the population by lateral connection strengths G_{ij}. The resulting subthreshold potential from the leaky integrator, u_i is passed through a threshold function g to give the neurons output a_i.

The learning rule for the $\phi_i(x, y)$ is the same as in the original algorithm. It is derived via direct gradient descent on E using the coefficient values $\hat{a}_i$ computed from the LCA. This yields the update rule

$$\Delta\phi_i(x, y) = \eta\, r(x, y)\, \hat{a}_i \tag{6}$$

where $r(x, y) = I(x, y) - \sum_i \hat{a}_i\, \phi_i(x, y)$ and η is the learning rate. Note that the equations given above for LCA are conditioned on the basis functions having unit length, so after each update they must be renomalized so that $\sum_{x,y} \phi_i(x, y)^2 = 1\ \ \forall_i$.

To summarize the above, the LCA computes for each image

$$\hat{\mathbf{a}} = \arg\min_{\mathbf{a}} E(\mathbf{I}, \mathbf{a}; \mathbf{\Phi}) \tag{7}$$

and the learning rule seeks to solve

$$\mathbf{\Phi}^* = \arg\min_{\mathbf{\Phi}} \langle E(\mathbf{I}, \hat{\mathbf{a}}; \mathbf{\Phi})\rangle \tag{8}$$

where the brackets $\langle\ \ \rangle$ denote average over many (millions) images.