**VS 265: Neural Computation** **Fall 2016**

# Linear time-invariant systems and convolution

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## 1 Linear, time-invariant systems

- Let us consider a dynamical system with input $x(t)$ and output $y(t)$:


- Such a system is said to be a linear, time-invariant system if it obeys the laws of superposition and scaling over time. That is, if you observe an output signal $y_1(t)$ in response to an input signal $x_1(t)$, and later observe an output $y_2(t)$ in response to input $x_2(t)$, then the response to the combination $\alpha x_1(t) + \beta x_2(t)$ is just $\alpha y_1(t) + \beta y_2(t)$.

- One way to characterize a linear, time-invariant system is by measuring its impulse response function. This is the response you would obtain to a small pulse of unit amplitude, $\delta(t)$, where

$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

  Let's call the output signal we measure in response to such an input $h(t)$.

- Once we have measured the impulse response function, $h(t)$, we can compute the output $y(t)$ in response to any complex input $x(t)$. The first step is to consider the signal $x(t)$ to be composed of a superposition of unit impulses of different amplitudes:

$$x(t) = \sum_j \delta(t - \tau_j)x(\tau_j)$$

  Now since the system is also time-invariant (i.e., it does not change its behavior over time), then we know that the response to a shifted impulse $\delta(t - \tau_j)$ is just $h(t - \tau_j)$. And so the response to a weighted sum of such shifted impulses is just a weighted sum of the resulting shifted impulse response functions. Or in the language of mathematics:

$$\delta(t) \to h(t)$$
$$\delta(t-\tau_j) \to h(t-\tau_j)$$
$$x(\tau_j)\delta(t-\tau_j) \to x(\tau_j)h(t-\tau_j)$$
$$\sum_j x(\tau_j)\delta(t-\tau_j) \to \sum_j x(\tau_j)h(t-\tau_j)$$

- Thus, the output $y(t)$ in response to the input signal $x(t)$ may be written as

$$y(t) = \sum_j x(\tau_j)h(t-\tau_j) \tag{1}$$

And in the limit that the spacing between time samples becomes infinitesimally small, this relation becomes exact and the sum turns into an integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{2}$$

# 2 Convolution

- The expression above is known as the convolution sum (1) or convolution integral (2). It tells us how to predict the output of a linear, time-invariant system in response to any arbitrary input signal.

- The mathematical shorthand notation for the convolution operation is to use the  symbol as follows:

$$y(t) = h(t) * x(t)$$

- One way of interpreting the convolution sum is just as we developed it above - i.e., it is simply a linear superposition of impulse response functions $h(t-\tau_j)$ each of which is multiplied by $x(\tau_j)$.

- The other (more common way) of interpreting the convolution sum is that it tells us that the output is computed by taking a weighted sum of the present and past input values. We can see this by writing out the sum in (1) above:

$$y(t) = h(0)x(t) + h(1)x(t-1) + h(2)x(t-2) + \dots$$

where we have assumed here for now that the times $\tau_j, \tau_{j+1}, \dots$ are spaced by one unit of time. Note also that we do not sum over values of $\tau_j$ for which $t - \tau_j < 0$. The reason is that for any physical system, $h(t)$ is defined only for $t > 0$. This indeed makes sense, because otherwise we would need to know future values of the input in order to compute the present output.

- Typically the impulse response function decays away with time, and there is a point at which we can consider it to be essentially zero and so we can truncate the expansion above at a certain number of taps (discrete samples).
- The convolution operation may also be thought of as a filtering operation on the signal $x(t)$, where the impulse response function $h(t)$ is acting as the filter. The shape of $h(t)$ determines which properties of the original signal $x(t)$ are filtered out. The design of filters is usually best thought of in the frequency domain, which we turn to next ...