Title: Convergence of map seeking circuits

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Convergence of map seeking circuits

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April 18, 2006

Abstract
The map seeking circuit is an explicit biologically motivated computational mechanism, which allows solution of problems in computer vision, image tracking, limb inverse kinematics and others. We formulate this algorithm as discrete dynamical system on a set $\Delta = \Pi_{i=1}^{L} \Delta^{(i)}$, where each $\Delta^{(i)}$ is a compact subset of a nonnegative orthant of $\mathbb{R}^n$, and show that for an open and dense set of initial conditions in $\Delta$ the corresponding solutions converge to either a vector with unique nonzero element in each $\Delta^{(i)}$ or to a zero vector. The first result implies that circuit finds a unique best mapping which relates reference pattern to target pattern; the second result is interpreted as “no match found”. These results verify numerically observed behaviour in numerous practical applications.

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1 Introduction
This paper studies a behavior of an algorithm using ideas and methods from dynamical systems theory. The algorithm, called a Map-Seeking Circuit (MSC), was developed by D. Arathorn [1] and has been applied by him to a variety of theoretical and practical problems in biological vision [2, 3, 4] and machine vision [5, 6], inverse kinematics and route planning [2, 7], and in cooperation with other investigators to dynamic image processing [12, 13], and high degree-of-freedom robotic motion control [14]. The MSC algorithm is applicable to a variety of inverse problems that can be posed as transformation-discovery problems, where the goal is to find the best transformation that maps a reference pattern to a target pattern.

The MSC algorithm was motivated by the structure and function of the cortical visual processing streams. A number of visual tasks such as stereo vision, determining shape from motion and shading and recognition of rigid and articulated objects can be posed as transformation-discovery problems and can be readily solved by MSC algorithm. The solution of more complex problems like object recognition involves a decomposition of an aggregate transformation into a sequence of component transformations. For example, the recognition of a known 3D object in an image which contains other objects (see Figure 1), involves discovering the transformations involved in image formation: the location of the projection of the object in the scene, the magnification of that projection and the orientation angles which produced the particular 2D projection of physical 3D object. For objects whose recognition requires determining interior surface shape rather than just occluding contour, lighting direction becomes an additional factor in the image formation transformations.
For objects which are not rigid, physical articulation or morphing transformations are composed with the image formation transformations and must be discovered in the process of recognition [4, 6].

The composition of transformations applies readily to inverse kinematics, in which the unknown transformations consist of a sequence of projections from the limb root via each limb segment to the target location for the end effector [2, 7, 14]. Similarly, for route finding and/or motion planning, the transformations to be discovered are the sequence of movements which will take the animal or robot from its current location and velocity to the target location and velocity [7, 14].

We illustrate the algorithm on a problem of recognition of a rigid 3D object in Figure 1. The problem, often referred to in machine and biological vision circles as the correspondence problem, is to identify the transformation which maps the model to the projection of the object in the scene, ignoring all the distracting objects in the scene. The model, or a target pattern, in this case is a 3D model of the surface of the pig defined by normal vectors located in space, Figure 1c. The 2D projection of the pig (reference pattern) appears in the input image Figure 1a. The MSC algorithm solves the inverse problem of finding the transformation that takes 2D image to the 3D model. It seeks the unknown transformation as a composition of (1) translation in the image plane, (2) rotation in the image plane, (3) scaling in the image plane, and (4) projections between 2D and 3D parameterized by azimuth and elevation. MSC arrives at a solution by a process of convergence which involves competitive culling of linear combinations (superpositions) of all the possible transformations. A graphical presentation of this process on the inverse of 3D-2D projections is seen in Figure 1d-f. This example ignores occlusion, background noise and image degradation, all of which are dealt with in [4, 5, 6].

The behavior of the convergence of MSC in discovering these transformations, regardless of the application, is the subject of this paper.

2 Results

We now describe the problem in more detail where we follow the exposition in [9]. Denote the reference (input) pattern by \(I\), denote the target (memory) pattern by \(M\), and let \(I, M\) lie in \(\mathbb{R}^p\). For a particular transformation \(T\) in a given class of transformations \(\mathcal{T}\) we define the correspondence associated with \(T\) to be

\[
\langle c(T) = \langle T(I), M \rangle \tag{1}\]

where \(\langle \cdot, \cdot \rangle \) denotes the inner product on \(\mathbb{R}^p\).

We assume that each \(T \in \mathcal{T}\) is a composition of \(L\) maps

\[
T = T^{(L)}_{i_L} \circ \ldots \circ T^{(2)}_{i_2} \circ T^{(1)}_{i_1} \tag{2}
\]

For each index \(\ell\) between 1 and \(L\), the maps \(T^{(\ell)}_{i_\ell}\) are taken from a collection of transformations which, by analogy to the layers in the visual cortex, will be called a layer. We also require each component transformation for layer \(\ell\), \(T^{(\ell)}_{i_\ell}\), to be linear and to be discretely indexed so that \(1 \leq i_\ell \leq n_\ell\). While linearity may seem to be a severe restriction, it holds in many important applications. For example, the component transformations in visual pattern recognition—translations, rotations, and rescalings—are each linear.

The task of maximizing the correspondence then reduces to selecting a particular transformation of the form (2) to maximize (1). Equivalently, one can select the indices \((i_1^*, i_2^*, \ldots, i_L^*)\) which maximize the correspondence array,

\[
c(i_1, i_2, \ldots, i_L) := \langle T^{(L)}_{i_L} \circ \ldots \circ T^{(2)}_{i_2} \circ T^{(1)}_{i_1}(I), M \rangle \tag{3}
\]

Hence one can solve the correspondence problem simply by constructing the \(N := n_1 \cdot n_2 \cdot \ldots \cdot n_L\) components of the \(L\)-dimensional array in (3) and then finding its maximum entry. For most of the interesting applications the number of components \(N\) is extremely large, so this approach is impractical.

A key idea in [2], which Arathorn refers to as ordering property of superpositions, allows the MSC algorithm to perform correspondence maximization iteratively with a cost per iteration that is proportional to the sum \(n_1 + n_2 + \ldots + n_L\). The idea is to embed the discretely parameterized linear transformations (2) in a family of continuously parameterized transformations. For each layer \(\ell\), take

\[
T^{(\ell)}_{x(i)} = \sum_{i=1}^{n_\ell} x_i^{(\ell)} T^{(\ell)}_{i} \tag{4}
\]
where $x_1^{(\ell)} \leq 1$ are gain coefficients. If we replace the individual maps $T_i^{(\ell)}$ in the right hand side of (3) by the linear combinations (4) we obtain the correspondence function

$$f(x^{(1)}, x^{(2)}, \ldots, x^{(L)}) := \langle T^{(L)}_{x^{(L)}}, \cdots, T^{(2)}_{x^{(2)}}, T^{(1)}_{x^{(1)}}(I), M \rangle$$

$$= \sum_{i_1=1}^{n_1} \cdots \sum_{i_L=1}^{n_L} c(i_1, \ldots, i_L)x^{(1)}_{i_1} \cdots x^{(L)}_{i_L}.$$  \hspace{1cm} (5)

The goal is to maximize the function $f$ on the set where $x_1^{(\ell)} \leq 1$ for all $\ell$.

We now describe the Map Seeking Circuit (MSC) algorithm [2]. We set

$$I^{(0)} := I \text{ and } I^{(\ell)} := T^{(\ell)}_{x^{(\ell)}}(I^{(\ell-1)}) := \sum_{i=1}^{n_1} x_i^{(\ell)} T_i^{(\ell)}(I^{(\ell-1)}),$$

where $I^{(\ell)}$ is a linear combination (superposition) of maps $T_i^{(\ell)}$ applied to the input to the $\ell$-th layer $I^{(\ell-1)}$. Similarly, we set

$$M^{(L)} := M \text{ and } M^{(\ell-1)} := T^{(\ell)}_{x^{(\ell)}}(M^{(\ell)}) := \sum_{i=1}^{n_1} x_i^{(\ell)} T_i^{(\ell)}(M^{(\ell)}),$$

Figure 1: MSC algorithm in image recognition. (a) input image; (b) edge filtered signal into layer 1 forward; (c) 3D surface normal model in memory; (d - f) convergence of superpositions of transformations, iterations 1, 8, and 25. (Figure taken from [3]; 3D models courtesy www.3DCafe.com).
where \( M^{(\ell)} \) is the backward input to the \( \ell \)-th layer and \( T^{(\ell)*} \) are Hermitian conjugates of maps \( T^{(\ell)} \). Therefore

\[
I^{(\ell)} := T^{(\ell)} \cdot \cdots T^{(2)}(I), \quad \ell = 1, \ldots, L, \\
M^{(\ell-1)} := T^{(\ell)*} \cdot \cdots T^{(L)}(M), \quad \ell = L, \ldots, 2, 1.
\]

Using (6) and (7), the objective function (5) can be expressed for any \( \ell = 1, \ldots, L \), as

\[
f(x^{(1)}, \ldots, x^{(L)}) = \langle I^{(\ell)}, M^{(\ell)} \rangle = \sum_{i=1}^{n_\ell} x_i^{(\ell)} \langle T_i^{(\ell)}(I^{(\ell-1)}), \ M^{(\ell)} \rangle.
\]

The first equality follows by taking adjoints in (5) and substituting (6) and (7). The second equality follows from (4) and the bilinearity of the inner product. Since (8) holds for at every layer \( \ell \) we can dynamical update the coefficients \( x_i^{(\ell)} \) synchronously on all layers. First a vector of matches

\[
L^{(\ell)} := \left( \langle T_1^{(\ell)}(I^{(\ell-1)}), M^{(\ell)} \rangle, \langle T_2^{(\ell)}(I^{(\ell-1)}), M^{(\ell)} \rangle, \ldots, \langle T_n^{(\ell)}(I^{(\ell-1)}), M^{(\ell)} \rangle \right)
\]

is computed. The greatest entry in this vector represents the best match between transformed input and the transformed memory. The weight \( x_i^{(\ell)} \) of the map \( T_i^{(\ell)} \) that produced the best match should be retained while other weights should be suppressed. Therefore we update the vector of gating coefficients \( x^{(\ell)} \) using a competition function \( C(\cdot) \)

\[
x^{(\ell)}(n+1) = C^{(\ell)}(x^{(\ell)}(n), L^{(\ell)}),
\]

where we set

\[
C^{(\ell)}(u, v) := \left\{ \begin{array}{ll}
\max(0, u - \kappa^{(\ell)}(1 - \frac{v}{\max(v)})) & \text{if } \max(v) \geq \epsilon^{(\ell)} \\
0 & \text{if } \max(v) < \epsilon^{(\ell)}
\end{array} \right.
\]

where \( \max(v) \) is the maximal component of the vector \( v \). The functions \( C^{(\ell)} \) for different \( \ell \) may differ in the choice of the constant \( \kappa^{(\ell)} \) and \( \epsilon^{(\ell)} \). However, the different choices of \( \epsilon^{(\ell)} \) do not significantly affect our argument and thus we simplify our bookkeeping by assuming \( \epsilon = \epsilon^{(\ell)} \) for all \( \ell = 1, \ldots, L \). Observe, that the function \( C^{(\ell)} \) preserves the value of the maximal weight \( x_i^{(\ell)} \) and lowers other weights \( x_j^{(\ell)} \) towards zero. If these weights are driven below the threshold \( \epsilon \) without convergence, they are all set to zero.

With the updated gating constants \( x^{(\ell)}(n+1) \) we compute updated values of \( I^{(\ell)} \) and \( M^{(\ell)} \) in (6) and (7) and iterate the whole process. We take the initial gating constants \( x_1^{(1)} \) to be equal to a small random perturbation of the value 1.

We now formulate the updates of the algorithm as iterations of a map on a compact space. Let \( \Delta^{(\ell)} := \{x^{(\ell)} \in \mathbb{R}^{n_\ell} \mid \sum x_j^{(\ell)} \leq nt \} \) and let

\[
\Delta = \Delta^{(1)} \times \ldots \times \Delta^{(L)}.
\]

The dynamics of each layer is described by

\[
x^{(\ell)}(n+1) := C^{(\ell)}(x^{(\ell)}(n), L^{(\ell)}(x^{(1)}(n), x^{(2)}(n), \ldots, x^{(L)}(n))),
\]

where \( n \) denotes the iteration number and where \( L^{(\ell)} : \mathbb{R}^{n_\ell} \to \mathbb{R}^{n_\ell} \) is the \( \ell \)-th layer transfer function (see (9)), defined by

\[
L_i^{(\ell)}(x^{(1)}, \ldots, x^{(L)}) = \langle T_i^{(\ell)}(I^{(\ell-1)}), M^{(\ell)} \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) is a dot product.

Let \( x = (x^{(1)}, \ldots, x^{(L)}) \) be the concatenation of vectors \( x^{(\ell)} \in \mathbb{R}^{n_\ell} \), let \( C \) be a concatenation of functions \( C^{(\ell)} \) and let \( L(x) \) denote the collection of layer transfer functions \( L^{(\ell)}(x) \). The dynamics of the whole circuit can be then expressed as

\[
x(n+1) := C(x(n), L(x(n))),
\]

where \( C, L : \mathbb{R}^{n_\ell} \to \mathbb{R}^{n_\ell} \) and \( n := \sum_{\ell=1}^{L} n_\ell \) is the total number of linear transformations in all \( L \) layers.. Let \( e_\ell \in \mathbb{R}^{n_\ell} \) be a vector whose \( i(\ell) \)-th coordinate is 1 and other coordinates are zero. We formulate our main result.
Theorem 2.1 Consider a discrete dynamical system generated on $\Delta$ by (13). Assume that $c(i_1,\ldots,i_n) \geq 0$ and set $c_{\min} := \min_{i \neq 0} c(i_1,\ldots,i_n)$, $c_{\max} := \max c(i_1,\ldots,i_n)$. Fix a set of constants $\kappa^{(1)}, \ldots, \kappa^{(L)}$ in competition functions $C^{(1)}, \ldots, C^{(L)}$ with the property $\kappa^{(\ell)} \leq \left( \frac{c_{\max}}{c_{\min}} \right)^2$ for all $\ell$.

Then for a generic correspondence array $\{c(i_1,\ldots,i_L)\}$ there is an open and dense set $G \subset \Delta$ with the following property. If initial condition $x(0) \in G$ then iterations $x(n)$ of (13) converge either to the zero vector or to a vector $(a_1 e_i^{(1)}, a_2 e_i^{(2)}, \ldots, a_L e_i^{(L)})$, for some positive numbers $a_1, \ldots, a_L$.

The expression “generic correspondence array $c(i_1,\ldots,i_L)$” means that there is an open and dense set in the space of all collections for which our results are true. The necessary condition for being in the generic set is that all elements of the collections are distinct, see Lemma 4.8 and Lemma 4.9, but it may not be sufficient (Lemma 4.10).

The assumptions for the main result are mild and are satisfied in all (known to us) implementations. The condition $c(i_1,\ldots,i_n) \geq 0$ is not very restrictive. Starting with an arbitrary set of coefficients $c(i_1,\ldots,i_L)$ we can satisfy the positivity condition by adding a constant to the nonzero coefficients.

The condition $\kappa^{(\ell)} \leq \left( \frac{c_{\max}}{c_{\min}} \right)^2$ relates the step size of the algorithm $\kappa^{(\ell)}$ to the set of coefficients $c(i_1,\ldots,i_L)$.

Note that the set $\Delta$ is a closed subset of $\mathbb{R}^{n+}$ with a non-empty boundary. As the algorithm eliminates weights $x_i^{(\ell)}$ by setting them to zero, it enters the boundary of $\Delta$. We can strengthen the result of Theorem 2.1 to state that the set $G$ is actually open and dense in majority of the boundary subsets of $\Delta$. Since the formulation of this results requires an additional notation, we have delegated its formulation to the Appendix (see Theorem 4.1).

We now outline the argument of the proof. We first characterize the fixed points of the map $C$ and then find a Lyapunov function for the map $C$. The key consequence of the existence of the Lyapunov function is Corollary 4.7, which shows that all solutions either converge to a point (see Theorem 4.1). Formulation of this result requires an additional notation, we have delegated it to the Appendix (see Theorem 4.1).

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by the translation mappings in layer 1. The circuit converges very quickly as can be seen in Figure 1, having found an approximate solution by iteration 8 and fully converged by iteration 25.

3 Discussion

The main result of this paper shows that for a generic correspondence array $c(i_1, \ldots, i_L)$ there is an open and dense set $G$ of initial gating constants $x_i^{(l)}$ such that the map-seeking circuit always converge to a either a zero solution (i.e. $x_i^{(l)} = 0$ for all layers $l$ and all mappings $i$) or it converges to a solution where on each layer $l$ there exists precisely one weight $x_i^{(l)}$ which is nonzero and all other weights are equal to zero. The first result is interpreted as "no match found", while the second implies that circuit finds unique composition

$$T = T_{i_L}^{(L)} \circ \ldots \circ T_{i_2}^{(2)} \circ \ldots \circ T_{i_1}^{(1)}.$$ 

This confirms numerical observations of the behaviour of the circuit. This result is independent of the choice of the set of mappings, as well as the reference and target patterns supplied. It is purely a function of the structure and the design of the algorithm.

Acknowledgement. The research of T. G. was partially supported by NSF-BITS grant 0129895, NIH-NCRR grant PR16445, NSF/NIH grant W0467 and NSF-CRCNS grant W0577.

4 Appendix

The Appendix is organized as follows. We first carefully define boundary subsets of $\Delta$ and formulate a stronger version of Theorem 2.1. In section 4.1 we characterize the fixed points of the map and in section 4.2 we find a Lyapunov function for the system. The key result of this section is Corollary 4.7, which shows that all solutions either converge to a point $(a_1e_1^{(1)}, \ldots, a_L e_1^{(L)})$ for some collection $a_L > 0$, to a zero solution, or to an internal fixed point. The sections 4.3 and 4.4 are devoted to an argument showing that for a generic MSC the set of initial conditions that converge to internal fixed points is nowhere dense.

The set $\Delta$ is a closed subset of $\mathbb{R}^{n+}$ with a non-empty boundary. As the algorithm eliminates weights $x_i^{(l)}$ by setting them to zero, it enters the boundary of $\Delta$, which we now describe. We define for each layer $l$ a non-empty collection of integers $\omega^{(l)} = (i_1, \ldots, i_{q(l)})$, $q(l) \geq 1$, $i_j \in \{1, \ldots, n_l\}$. For any such $\omega^{(l)}$ we denote by $\mathbb{R}^{n+}_{\omega^{(l)}}$, the boundary part of $\mathbb{R}^{n+}$ consisting of vectors of the form $(0, u_1, 0, \ldots, u_{q(l)}, 0)$, where the nonzero elements are in positions $i_1, \ldots, i_{q(l)}$. We set

$$\Delta_{\omega^{(l)}} := \mathbb{R}^{n+}_{\omega^{(l)}} \cap \Delta^{(l)}.$$ 

Similarly, for $\Omega = (\omega^{(1)}, \ldots, \omega^{(L)})$ we denote $\mathbb{R}^{n+}_{\Omega}$ the boundary part of $\mathbb{R}^{n+}$ consisting of vectors of the form $(u^{(1)}, \ldots, u^{(L)})$ such that $u^{(l)} = (0, u_1^{(l)}, 0, \ldots, u_{q(l)}, 0)$, where the nonzero elements of $u^{(l)}$ are in positions specified by $\omega^{(l)} = (i_1^{(l)}, \ldots, i_{q(l)}^{(l)})$. We set

$$\Delta_{\Omega} := \mathbb{R}^{n+}_{\Omega} \cap \Delta = \Pi_{l=1}^{L} \Delta_{\omega^{(l)}}.$$ 

Let $q = \sum_{l=1}^{L} q(l)$ be the total sum of nonzero components in $\mathbb{R}^{n+}_{\Omega}$. We define the set

$$\Xi := \{ \Omega = (\omega^{(1)}, \ldots, \omega^{(L)}) \mid \text{there are at least two layers } l, r \text{ with } |\omega^{(l)}| \geq 2, |\omega^{(r)}| \geq 2 \}.$$ 

(A.1)

Note that the space $\mathbb{R}^{n+}_{\Omega}$ with $\Omega \notin \Xi$ is a product of half-lines and at most one set isomorphic to the positive cone of $\mathbb{R}^{2}$. We now formulate an extension of Theorem 2.1

Theorem 4.1 Assume all assumptions of Theorem 2.1.

Then for a generic correspondence array $\{c(i_1, \ldots, i_L)\}$ there is an open and dense set $G \subset \Delta$, which is also open and dense in every boundary set $\Delta_{\Omega}$ for all $\Omega \in \Xi$, with the following property. If initial condition $x(0) \in G$ then iterations $x(n)$ of (13) converge either to the zero vector or to a vector $(a_1e_1^{(1)}, a_2e_2^{(2)}, \ldots, a_L e_L^{(L)})$, for some positive numbers $a_1, \ldots, a_L$. 


4.1 Fixed points of $C$

We start with a technical Lemma that provides the bridge between the MSC algorithm and the correspondence function $f$ in (5). Fix $\Omega \in \Xi$ and let $L_{\Omega}$ and $f_{\Omega}$ be restrictions of the functions $L$ and $f$ to the set $\mathbb{R}_{n+}^{n}$.

Lemma 4.2 ([9, 8]) The function $L$ is the gradient of the cost function $f$

\[ [L^{(\ell)}]_{i}(x^{(1)}, \ldots, x^{(L)}) = [\nabla^{(\ell)} f_{\Omega}], \]

where

\[ \sum_{i_{1} \in \omega^{0}} \ldots \sum_{i_{L} \in \omega^{0}} c(i_{1}, \ldots, i_{\ell}, \ldots, i_{L})x_{i_{1}}^{(1)} \ldots x_{i_{\ell}}^{(1)} \ldots x_{i_{L}}^{(1)} \]

Further,

\[ [L^{(\ell)}(x)]_{i} > 0 \quad \text{if} \quad i \in \omega^{(\ell)} \quad \text{and} \quad [L^{(\ell)}(x)]_{i} = 0 \quad \text{if} \quad i \notin \omega^{(\ell)}. \]

Proof. Differentiating (8) we obtain the components of the gradient:

\[ \frac{\partial f}{\partial x_{i}^{(\ell)}} = (T_{i}^{(\ell)}(J^{(\ell-1)}), \quad M^{(\ell)} = [L^{(\ell)}]_{i}(x^{(1)}, \ldots, x^{(L)}). \]

Restriction to the subset $\Omega$ finishes the first result. To show the second part we observe that the right hand side of (A.2) is positive since all $x_{i_{\ell}}^{(\ell)} > 0$ and the coefficients $c(i_{1}, \ldots, i_{L}) > 0$. The result follows. \hfill \Box

Lemma 4.3 A point $x \in \mathbb{R}_{n+}^{n+}$, $x \neq 0$ is a fixed point of the map (13), if, and only if,

\[ L(x) = (a_{1}1_{\omega}^{(1)}, \ldots, a_{L}1_{\omega}^{(1)}), \]

where $1_{\omega}^{(\ell)} := e_{i_{1}}^{(\ell)} + e_{i_{2}}^{(\ell)} + \ldots + e_{i_{\ell}}^{(\ell)}$ if $i_{j} \in \omega^{(\ell)}$, be the vector of 1’s in all directions in $\omega^{(\ell)}$.

Furthermore, every point of the form $e = (a_{1}e_{i_{1}}^{(1)}, a_{2}e_{i_{2}}^{(2)}, \ldots, a_{L}e_{i_{L}}^{(L)})$ for any positive constants $a_{i}$, is a fixed point of the map $C$.

Proof. Take $x \in \mathbb{R}_{n+}^{n+}$ with $x \neq 0$. Then $x$ is a fixed point if it satisfies $x = C(x, L(x))$. From the form of the competition functions $C^{(\ell)}$ follows that for all $\ell$ we must have $L_{\ell}(x) = K^{(\ell)} = \max L^{(\ell)}(x)$ for all $i$ where $L_{\ell}(x) > 0$. Thus $L^{(\ell)}(x)$ has the form above with $a_{\ell} = K^{(\ell)}$.

To show the second part take $\Omega = (\omega^{(1)}, \ldots, \omega^{(L)})$ and each $\omega^{(\ell)} = \{m_{\ell}\}$ contains exactly one element. Then a nonzero $x \in \mathbb{R}_{n}$ has the form $e = (a_{1}e_{i_{1}}^{(1)}, \ldots, a_{L}e_{i_{L}}^{(L)})$. Applying (A.2) to such $e$ we get

\[ [L^{(\ell)}]_{i}(e) = (\Pi_{i_{\ell} \neq m_{\ell}})c(m_{1}, \ldots, m_{L}) \quad \text{for} \quad i = m_{\ell} \quad \text{and} \quad [L^{(\ell)}]_{i}(e) = 0 \quad \text{if} \quad i \neq m_{\ell}. \]

Since for each $\ell$ the vector $L^{(\ell)}_{\Omega}$ has a single non-zero element, its maximum is achieved at such element and $K^{(\ell)} = (\Pi_{i_{\ell} \neq m_{\ell}})c(m_{1}, \ldots, m_{L})$. By the part one of this Lemma $e$ is a fixed point of $C$. \hfill \Box

Lemma 4.4 Let $u = C(x)$. Let $\zeta^{(\ell)}(x) := \{i \mid x_{i}^{(\ell)} \neq 0\}$ be the set of indices of nonzero elements of $x^{(\ell)}$ and let $\eta^{(\ell)}(x) := \{i \mid C_{i}^{(\ell)}(x) \neq 0\}$ be the set of indices of nonzero elements of $C^{(\ell)}(x)$. Then for all $\ell = 1, \ldots, L$

$$ \eta^{(\ell)}(x) \subset \zeta^{(\ell)}(x) $$
Proof. We will consider each part of the function $C$ separately. Recall that $K(ℓ) = \max_{L(ℓ)}(x)$. Fix $ℓ$ and rewrite the $i$-th component of the function $C(ℓ)$ as

$$C_i(ℓ)(x^,(ℓ), L^,(ℓ)(x))) = \max(0, x_i^,(ℓ) - \kappa(ℓ)(1 - \frac{[L_i^,(ℓ)(x)]}{K(ℓ)})$$

$$= \begin{cases} 0 & \text{if } x_i^,(ℓ) \leq \kappa(ℓ)(1 - \frac{[L_i^,(ℓ)(x)]}{K(ℓ)}) \\ x_i^,(ℓ) - \kappa(ℓ)(1 - \frac{[L_i^,(ℓ)(x)]}{K(ℓ)}) & \text{if } x_i^,(ℓ) \geq \kappa(ℓ)(1 - \frac{[L_i^,(ℓ)(x)]}{K(ℓ)}) \end{cases} \quad (A.4)$$

Assume that $x_i^,(ℓ) = 0$. Then $\kappa(ℓ)(1 - \frac{[L_i^,(ℓ)(x)]}{K(ℓ)}) \geq 0$, since $K(ℓ) = \max L(ℓ)(x)$. Since $x_i^,(ℓ) = 0$, this implies $x_i^,(ℓ) \leq \kappa(ℓ)(1 - \frac{[L_i^,(ℓ)(x)]}{K(ℓ)})$. Therefore by (A.4) we have $u_i^,(ℓ) = C_i(ℓ)(x) = 0$.

\[ \square \]

Corollary 4.5 For any $Ω$ the boundary set $IR^(+)^n$ is positively invariant under the map $C$.

4.2 The Lyapunov function

The key result is a construction of a Lyapunov function [10, 11].

Lemma 4.6 If $u = (u(1), \ldots, u(L)) \in IR^(+)^n$ then

1. $|C(u, L(u))| \leq |u|$, where $|v| = \sum_i v_i$ denotes the sum of the elements of the vector $v$;

2. The previous inequality is strict, unless $u$ is a fixed point of $C$.

Proof. Observe that by (A.4) the set $η^,(ℓ)(x)$ is the set of indices where $x_i^,(ℓ) \geq \kappa(ℓ)(1 - \frac{[L_i^,(ℓ)(x)]}{K(ℓ)})$.

We compute the sum of elements of the vector $C^,(ℓ)(x)$:

$$|C^,(ℓ)(x^,(ℓ), L^,(ℓ)(x)))| = \sum_{i \in η^,(ℓ)(x)} x_i^,(ℓ) - \kappa(ℓ)(1 - \frac{[L_i^,(ℓ)(x)]}{K(ℓ)})$$

$$\leq \sum_{i \in η^,(ℓ)(x)} x_i^,(ℓ)$$

$$\leq \sum_{i \in ζ^,(ℓ)(x)} x_i^,(ℓ)$$

$$= |x^,(ℓ)|.$$  

Here the first inequality holds because the maximum of rescaled vector $\frac{L_i^,(ℓ)(x)}{K(ℓ)}$ is 1. The second inequality follows from Lemma 4.4. Since $x$ is a concatenation of vectors $x^,(ℓ)$ and $|x| = |x^,(1)| + \ldots + |x^,(L)|$ by the definition of $|⋅|$, this proves the first part of the Lemma.

The equality $|C(x, L(x))| = |x|$ happens when for all $ℓ = 1, \ldots, L$, both of the inequalities above are in fact equalities. For a fixed $ℓ$ the equality $|C^,(ℓ)(x, L^,(ℓ)(x)))| = |x^,(ℓ)|$ implies that, first, for all $i \in η^,(ℓ)(x^,(ℓ))$ we have $[L_i^,(ℓ)(x)] = K(ℓ)$, and, second, that $ζ^,(ℓ)(x) = η^,(ℓ)(x)$.

Since this holds for every $ℓ = 1, \ldots, L$, $x$ is a fixed point of $C$ by Lemma 4.3. \[ \square \]

Corollary 4.7 1. For all $Ω$ the set $∆_Ω$ is positively invariant (i.e. $C(∆_Ω) ⊂ ∆_Ω$) under the map (13).

2. There is an integer $N$ such that for any initial condition $x(0) ∈ Δ$, then the $N$-th iterate $x(N)$ is either

(a) $x(N) = 0$; or

(b) $x(N) = e = (a_1 e^{(1)}_1, \ldots, a_L e^{(L)}_L)$ for some collection of $a_i > 0$; or
(c) the trajectory \( \{ x(k) \}_{k=1}^{\infty} \rightarrow u \) where \( L(u) = (a_1^{(1)}, \ldots, a_L^{(1)}) \) is a fixed point with at least one \( \ell \) with \( |\omega(\ell)| \geq 2 \).

**Proof.** The first statement is a corollary of Lemma 4.6 and Corollary 4.5.

To show the second part, observe that the number of nonzero components \( c^{(\ell)}(x) \) is a non-increasing function by Corollary 4.5. Since this function has also discrete set of values, it must be eventually constant. Let \( N(x) \) be such that for all \( n \geq N \) the number of nonzero components \( c^{(\ell)}(x(n)) \) of \( x(n) \) is constant. Since \( N(x) \) depends continuously on \( x \) and \( \Delta \) is compact, there exists a uniform \( N \) valid for all \( x \in \Delta \). If \( c^{(\ell)}(x(N)) \) has a single component for all \( \ell \) then by Corollary 4.5 \( x(N) = e \), which satisfies (b). It follows from the form of \( f \) (see 5) and Lemma 4.2 that if there is an \( \ell \) such that \( |c^{(\ell)}(x(N))| = 0 \) then \( x(N) = 0 \).

Finally, assume there is an \( \ell \) such that \( |c^{(\ell)}(x(N))| = s \geq 2 \) and \( |c^{(\ell)}(x(n))| \geq 1 \) for \( i \neq \ell \). Then \( |c^{(\ell)}(x(k))| = s \) for all \( s \geq N \). Since the Lyapunov function is bounded below by zero, we must have that \( x(k) \rightarrow u \) and \( |C(u, L(u))| = |u| \). Further, by continuity we have \( |c^{(\ell)}(u)| = s \geq 2 \). By Lemma 4.6.2 \( x(n) \) converges to a fixed point which by Lemma 4.3 has the advertised form. \( \square \)

### 4.3 Internal fixed points

As a consequence of Corollary 4.7, to prove Theorem 2.1 we need to show that there is an open and dense set \( W \) of initial conditions \( x(0) \), such that the iterations \( x(n) \) do not converge to a fixed point \( x \) satisfying (c) of the Corollary 4.7. Then the proof of Theorem 2.1 will follow from Lemma 4.6 and Corollary 4.7.

The fixed points \( u \) of \( C \) which satisfy condition (c) above will be called internal fixed points, since there must be at least one layer \( \ell \) where \( u^{(\ell)} \) is in the interior of \( \Delta_{\omega(\ell)} \). We now look more closely at these internal fixed points. Recall that \( \omega^{(\ell)} = (i_1, \ldots, i_{q(\ell)}) \), \( i_j \in \{1, \ldots, n_{\ell} \} \) is a non-empty collection of integers and \( \Omega = (\omega^{(1)}, \ldots, \omega^{(L)}) \) is a collection of \( \omega^{(\ell)} \). Let

\[
Z_{\Omega} := \{ x \in \mathbb{R}^{n^+} | x = (a_1^{1}, \ldots, a_L^{1}) \}
\]

and

\[
B_{\Omega} = \{ x \in \mathbb{R}^{n^+} | L(x) \in Z_{\Omega} \}.
\]

Next Lemma justifies the definition of the class \( \Xi \) of \( \Omega \)'s (see A.1), since only the boundary sets \( \Delta_{\Omega} \) with \( \Omega \in \Xi \) may contain internal fixed points.

**Lemma 4.8** Assume that the collection \( c(i_1, \ldots, i_{q(\ell)}) \) has distinct elements. If \( \Omega = (\omega^{(1)}, \ldots, \omega^{(L)}) \) has a unique \( \omega^{(\ell)} \) such that \( |\omega^{(\ell)}| = 2 \) and \( |\omega^{(i)}| = 1 \) for all \( i \neq \ell \), then \( B_{\Omega} = \emptyset \).

**Proof.** By assumption there is a unique layer \( \ell \) such that \( |\omega^{(\ell)}| = 2 \). Assume without loss that \( \omega^{(\ell)} = \{1, 2\} \). Since \( L(x) \in Z_{\Omega} \) implies \( L_1^{(\ell)}(x) = L_2^{(\ell)}(x) \), by (A.2) we get

\[
L_1^{(\ell)}(x) = c(m_1, m_2, \ldots, 1, \ldots, m_L)x_{m_1}^{(1)}x_{m_2}^{(2)} \ldots \hat{x}^{(\ell)}_ix_{m_i}^{(L)} \ldots x_{m_L}^{(L)} = c(m_1, m_2, \ldots, 2, \ldots, m_L)x_{m_1}^{(1)}x_{m_2}^{(2)} \ldots \hat{x}^{(\ell)}_ix_{m_i}^{(L)} \ldots x_{m_L}^{(L)} = L_2^{(\ell)}(x)
\]

where notation \( \hat{x}^{(\ell)}_i \) indicates that there are no \( x_i^{(\ell)} \) in the expression. This implies \( c(m_1, m_2, \ldots, 1, \ldots, m_L) = c(m_1, m_2, \ldots, 2, \ldots, m_L) \), contradicting our assumption. \( \square \)

Let

\[
Z := \bigcup_{\Omega \in \Xi} Z_{\Omega}, \quad (A.5)
\]

where the union is over the collection of all \( \Omega \in \Xi \). By Lemma 4.3 and Corollary 4.7 the set

\[
Fix := \{ x \in \mathbb{R}^{n^+} | L(x) \in Z \} \quad (A.6)
\]

is the set of internal fixed points of the map \( C \).
Lemma 4.9 If the collection \( c(i_1, \ldots, i_L) \) has distinct elements, then the set \( \text{Fix} \) is closed and nowhere dense in \( \mathbb{R}^{n+} \) and the intersection \( \text{Fix} \cap \mathbb{R}^{n+}_\Omega \) is closed, nowhere dense subset of \( \mathbb{R}^{n+}_\Omega \), for every \( \Omega \in \Xi \).

Proof. We first observe that since the function \( L \) is continuous and \( Z \) and \( \mathbb{R}^{n+}_\Omega \) are closed, the set \( \text{Fix} \cap \mathbb{R}^{n+}_\Omega \) is closed for each \( \Omega \).

We now prove the density of the complement of \( \text{Fix} \) in every \( \mathbb{R}^{n+}_\Omega \) with \( \Omega \in \Xi \). Fix \( \Omega \) and assume, contrary to our assertion, that there is an open set \( D \subset \mathbb{R}^{n+}_\Omega \) such that \( L(D) \subset Z \). By definition of \( Z \) this means that there exists \( \Omega' \) with \( \Omega' \in \Xi \) such that \( L(D) \subset \mathbb{Z}_{\Omega'} \), that is, for all \( \ell = 1, \ldots, L \),

\[
L^{(\ell)}(D) \subset \mathbb{Z}_{\Omega'}^{(\ell)}.
\]

Choose \( \ell \) with \(|\omega^{(\ell)}| \geq 2 \). Then there must exist two coordinates \( i, j \in \{1, \ldots, n_\ell\} \) such that

\[
L_i^{(\ell)}(D) = L_j^{(\ell)}(D).
\]

By (A.2) this is equivalent to

\[
\sum_{\omega(k) \neq \omega^{(\ell)}} c(i_1, \ldots, i_\ell)x_1^{(1)} \ldots x_\ell^{(L)} = \sum_{\omega(k) \neq \omega^{(\ell)}} c(i_1, \ldots, j, i_\ell)x_1^{(1)} \ldots x_\ell^{(L)}
\]

for all \( x \in D \). Since \( D \) is open, we have \( c(i_1, \ldots, i, i_\ell) = c(i_1, \ldots, j, i_\ell) \) for all \( i_k \in \omega(k) \) with \( \omega(k) \neq \omega^{(\ell)} \). This contradicts our assumption and finishes the proof of the Lemma.

Now we show that every internal fixed point \( x \in \text{Fix} \) is unstable, i.e. it has at least one eigenvalue with modulus greater than 1. Notice, that since the function \( L \) has the form described in (A.2), it has the following scaling property. If \( x = (x^{(1)}, \ldots, x^{(L)}) \) and \( y = (b_1x^{(1)}, \ldots, b_Lx^{(L)}) \) then

\[
L^{(\ell)}(y) = (\Pi_{j \neq \ell} b_j) L^{(\ell)}(x)
\]

for all \( \ell \). We call \( \lambda := (b_1, \ldots, b_L) \), where all \( b_j > 0 \), a multi-scaling factor and write \( y = \lambda x \). With this notation, if \( x \) is an internal fixed point that satisfies \( L(x) = (a_11_{\omega^{(1)}}, \ldots, a_L1_{\omega^{(L)}}) \) then

\[
L(\lambda x) = (a_1(\Pi_{j \neq 1} b_j)1_{\omega^{(1)}}, \ldots, a_L(\Pi_{j \neq L} b_j)1_{\omega^{(L)}})
\]

and hence it is again an internal fixed point. Let

\[
\mathbf{u} := \{ v \in \Delta \mid v = \lambda u \text{ for some multi-scaling factor } \lambda \}
\]

be the set of all fixed points related to \( u \) by scaling. We see that each internal fixed point belongs to an \( L \) dimensional cone-like space of fixed points. Therefore a linearization at each internal fixed point has eigenvalue 1 with a multiplicity at least \( L \). However, for any \( \Omega \in \Xi \) the set \( \Delta_\Omega \subset \mathbb{R}^{n+}_\Omega \) has dimension \( q \geq L + 2 \). Therefore for a generic \( c(i_1, \ldots, i_L) \) the set \( \text{Fix} \cap \mathbb{R}^{n+}_\Omega \) should have codimension 2. The proof of the next Lemma is based on the fact that for a generic collection \( \{c(i_1, \ldots, i_L)\} \) there is always at least one eigenvalue with the modulus greater then 1, and one eigenvalue with the modulus smaller then 1.

Lemma 4.10 There exists an open and dense set of coefficients \( c(i_1, \ldots, i_L) \) such that for all \( \Omega \in \Xi \) and all internal fixed points \( u \in \text{Fix} \cap \mathbb{R}^{n+}_\Omega \), satisfying \( L(u) = (a_11_{\omega^{(1)}}, \ldots, a_L1_{\omega^{(L)}}) \) with \( \Omega = (\omega^{(1)}, \ldots, \omega^{(L)}) \), the set

\[
W_\Omega(u) := \{ x \in \mathbb{R}^{n+} \cap \mathbb{R}^{n+}_\Omega \mid \lim_{n \to \infty} C^n(x) = \lambda u \text{ for some } \lambda \}
\]

is closed and nowhere dense in \( \Delta_\Omega \).

Proof. Take a point \( u \in \text{Fix} \) with \( L(u) = (a_11_{\omega^{(1)}}, \ldots, a_L1_{\omega^{(L)})} \) set \( \Omega = (\omega^{(1)}, \ldots, \omega^{(L)}) \), and assume a sequence of iterates \( \{x(n)\}_{n=1}^\infty \subset \mathbb{R}^{n+}_\Omega \), \( x(n + 1) = C(x(n)) \), converges to \( u \). We assume without loss of generality that \( \Omega = (\omega^{(1)}, \ldots, \omega^{(L)}) \) where \( \omega^{(\ell)} = (1, 2, 3, \ldots, q(\ell)) \) for all \( \ell = 1, \ldots, L \). To each point \( x(n) \) we can assign a collection \( \alpha(n) = (m_1, m_2, \ldots, m_L), m_\ell \in \omega^{(\ell)} \), with the property that \( L_\ell^{(\ell)}(x(n)) \) has the
maximal element $L^{(ℓ)}_{m_ℓ}(x(n))$. Since the number of distinct collections $α$ is finite, there is a subsequence $x(k_n)$ with $k_n → ∞$ such that $α(x(k_n))$ is constant. We rename the subsequence to be again $\{x(n)\}$.

By Corollary 4.5 the space $ℝ^{n+}_Ω$ is positively invariant. Let $C_Ω : ℝ^{n+}_Ω → ℝ^{n+}_Ω$ be the restriction of the map $C$ to $ℝ^{n+}_Ω$ defined by

$$u_i^{(ℓ)} := C^{(ℓ)}_{iℓ}(x) = X_i^{(ℓ)} + κ^{(ℓ)}(\frac{[L^{(ℓ)}(X)]_i}{K^{(ℓ)}} - 1),$$

where $X := (X^{(1)},...,X^{(L)})$, $X^{(ℓ)} = (x_1^{(ℓ)},...,x_q^{(ℓ)},0,0,...,0)$, and $K^{(ℓ)}$ is the maximal element of the vector $L^{(ℓ)}(x)$. Similarly, we will denote by $L^{(ℓ)}_Ω$ the restriction of $L$ to $ℝ^{n+}_Ω$.

Since $x(n) → u$ the sequence of unit vectors

$$v(n) := \frac{C(x(n+1)) - C(x(n))}{|C(x(n+1)) - C(x(n))|}$$

converges to an eigenvector $v$ of the derivative matrix $\frac{dC_Ω}{du}(u)$ with the corresponding eigenvalue with the modulus less or equal to 1. The derivative matrix $\frac{dC_Ω}{du}(u)$ is a $q × q$ matrix of the form $I + A$, where $I$ is the $q × q$ identity matrix and $A$ is a block matrix with $l × l$ blocks, where $(ℓ,s)$-block, $ℓ = 1, ..., L$, $s = 1, ..., L$, has the size $q(ℓ) × q(s)$. The $(i,t)$ element of the $(ℓ,s)$ block of $A$ is

$$[A(u)]^{(ℓ)}_{i,t} = \frac{κ^{(ℓ)}}{(L^{(ℓ)}_Ω)^2} \left( \frac{∂L^{(ℓ)}_i}{∂x^i} - \frac{∂L^{(ℓ)}_{m_ℓ}}{∂x^i} \right), \quad (A.8)$$

for all $1 ≤ i ≤ q(ℓ)$ and $1 ≤ t ≤ q(s)$. Notice that this is well defined since the sequence of vectors $L^{(ℓ)}(x(n))$ has the same maximal element $L^{(ℓ)}_{m_ℓ}(x(n))$ for all $ℓ$. By (A.2) each $(ℓ,ℓ)$ block of the matrix $A$ is zero. The trace of $I + A$ is therefore $q = \sum_{ℓ=1}^{L} q_ℓ$, which is the sum of all eigenvalues. Since there are $q$ eigenvalues, either all eigenvalues are equal to 1, or there is a pair of eigenvalues $λ_1, λ_2$ with $|λ_1| > 1$ and $|λ_2| < 1$. Thus all we need to show is that for all $Ω ∈ Ξ$, not all eigenvalues of $\frac{dC_Ω}{du}(x)$ are equal to 1. Assume to the contrary, that all eigenvalues of $I + A$ are equal to 1. Then by the Jordan normal form it follows that $A$ is nilpotent, i.e. there exists a power $N$ such that $A^N$ is the zero matrix. For any $Ω ∈ Ξ$ consider the corresponding matrix $A = A_Ω(u)$, where we emphasize the dependence of the matrix $A$ on both $Ω$ and the internal fixed point $u$. The proof of the Lemma will be complete if we prove the following claim, since it implies that the matrix $A_Ω(u)$ is not nilpotent and thus not all eigenvalues of $I + A$ are on the unit circle.

Claim 4.11 For an open and dense set of coefficients $c(i_1, ..., i_L)$ there is a nonzero diagonal element $a^{(ℓ)}_{i_ℓ}(u)$ of the matrix $A_Ω^{(ℓ)}(u)$.

Before we prove the claim, observe that

1. each diagonal $(ℓ,ℓ)$ block of $A_Ω$ is zero; and
2. for each $ℓ$ the $m(ℓ)$-th row is zero by the formula (A.8).

This second fact implies that for each $ℓ$ with $|ω_ℓ| = 1$, all corresponding blocks $(ℓ,s)$ for all $s$ are zero. Observe, that this implies that $I + A$ has at least $L$ eigenvalues 1. These correspond to the directions along the family of internal fixed points related by a multi-factor scaling. This opens a real possibility that the matrix $A_Ω$ may be nilpotent. Define $A_Ω$ to be the matrix which has 1 in each position of $A$, that is different then positions forced to be zero by (1) and (2) above. The following result shows that the form of the matrix $A_Ω$ for $Ω ∈ Ξ$ is compatible with $A_Ω$ being not nilpotent.

Claim 4.12 For all $Ω ∈ Ξ$, there is a diagonal element of the matrix $A^2$ that is non-zero.

Proof. Since $Ω ∈ Ξ$, there are $ℓ$ and $s$ such that $ω_ℓ$ and $ω_s$ have at least two elements. Recall that $m_ℓ$ denotes the index of the maximal element of $L^{(ℓ)}$, that is

$$L^{(ℓ)}_{m_ℓ} = \max_i L^{(ℓ)}_i.$$
Take $i \neq m_t$ and $j \neq m_s$ and consider $[\bar{a}_{ii}^{(\ell)}]^{(\ell)}$ the $(i, i)$-th element of the matrix $\bar{A}^2$ in the $(\ell, \ell)$ block. Since both the $(i, j)$ element of the $(\ell, s)$ block and the $(j, i)$ element of the $(s, \ell)$ block of $A$ are 1, $[\bar{a}_{ii}^{(\ell)}]^{(\ell)} \neq 0$.

**Proof of Claim 4.11** We consider the term $[a_{ii}^{(\ell)}]^{(\ell)}$ at the same position as the non-zero term $[\bar{a}_{ii}^{(\ell)}]^{(\ell)}$ in the previous Claim. To simplify notation we will use $a_{ii}^{(\ell)}$ to denote this term. We start with a formula for $a_{ii}^{(\ell)}$ which follows from (A.8):

$$a_{ii}^{(\ell)}(u) = \sum_{s, t} \frac{\kappa^{(\ell)}}{(L_m^{(s)})^2} \left( \frac{\partial L_i^{(s)}}{\partial x_t} L_t^{(s)} - \partial L_i^{(s)} \right) \left( \frac{\partial L_i^{(s)}}{\partial x_t} L_t^{(s)} - \partial L_i^{(s)} \right)$$

$$= \sum_{s, t} \frac{\kappa^{(\ell)}}{L_m^{(s)}} \left( \frac{\partial L_i^{(s)}}{\partial x_t} x_t^{(i)} - \partial x_t^{(i)} \frac{\partial L_i^{(s)}}{\partial x_t} \right) \left( \frac{\partial L_i^{(s)}}{\partial x_t} x_t^{(i)} - \partial x_t^{(i)} \frac{\partial L_i^{(s)}}{\partial x_t} \right)$$

(A.9)

where the second equality follows from the assumption that $L(u) = (a_1 1_{\omega(1)}, \ldots, a_L 1_{\omega(1)})$ and thus $L_m^{(s)} = L_i^{(\ell)}$ and $L_m^{(s)} = L_i^{(\ell)}$, whenever these values are nonzero. We compute the functions in (A.9) using (A.2)

$$L_m^{(s)}(x) = \sum_{\omega(s) \neq \omega^{(\ell)}} c(i_1, \ldots, i_{\ell-1}, m_t, i_{\ell+1} \ldots i_L) x_{i_1}^{(1)} \ldots \hat{x}^{(\ell)} \ldots x_{i_L}^{(L)}$$

(A.10)

$$\frac{\partial L_i^{(s)}}{\partial x_t^{(i)}}(x) = \sum_{\omega(s) \neq \omega^{(\ell)}} c(i_1, \ldots, i_{\ell-1}, i, i_{\ell+1} \ldots i_L) x_{i_1}^{(1)} \ldots \hat{x}^{(\ell)} \ldots x_{i_L}^{(L)}$$

where we use notation $\hat{x}^{(\ell)}$ to denote the fact that the variables $x^{(\ell)}$ are missing in a given expression.

We multiply all elements $c(i_1, \ldots, c_t, \ldots, i_L)$ with $c_t \neq m_t$ by a constant $b$ and observe how the function $a_{ii}^{(\ell)}(u, b)$ behaves under such scaling. Since $a_{ii}^{(\ell)}(u, b)$ is an analytic function of $b$, it is either identically zero, or, except for a finite number of exceptional values of $b$, we have $a_{ii}^{(\ell)}(u, b) \neq 0$. Observe that the functions $L_m^{(s)}$ and $\frac{\partial L_i^{(s)}}{\partial x_t^{(i)}}$ do not contain $c(i_1, \ldots, c_t, \ldots, i_L)$ with $c_t \neq m_t$. On the other hand every summand in functions $L_m^{(s)}$, $\frac{\partial L_i^{(s)}}{\partial x_t^{(i)}}$ and $\frac{\partial L_i^{(s)}}{\partial x_t^{(i)}}$ is being scaled by $b$. We write (A.9) as

$$a_{ii}^{(\ell)}(u, b) = \sum_{s, t} \kappa^{(\ell)} \left( \frac{\partial L_i^{(s)}}{\partial x_t^{(i)}} L_t^{(s)} - \partial x_t^{(i)} \frac{\partial L_i^{(s)}}{\partial x_t^{(i)}} \right) \left( \frac{\partial L_i^{(s)}}{\partial x_t^{(i)}} L_t^{(s)} - \partial x_t^{(i)} \frac{\partial L_i^{(s)}}{\partial x_t^{(i)}} \right)$$

Under the above scaling, the second term remains unchanged since both the numerator and the denominator are scaled by $b$, while in the first term only one term in the numerator is scaled

$$b \frac{\partial L_i^{(s)}}{\partial x_t^{(i)}} \frac{\partial L_i^{(s)}}{\partial x_t^{(i)}}$$

Since $\frac{\partial L_i^{(s)}}{\partial x_t^{(i)}} \neq 0$ by the same argument as in Lemma 4.2, for $b >> 1$ we get that $a_{ii}^{(\ell)}(u, b) \neq 0$. Therefore there is an open and dense set of values of $b$ for which for which $a_{ii}^{(\ell)}(u, b) \neq 0$. Since the function $L$ is multi-linear, we also have $a_{ii}^{(\ell)}(\lambda u, b) \neq 0$ for any multi-scaling factor $\lambda$ and the same $b$. Furthermore by continuity and for a fixed $b$, there is an open set of $y$ in a neighbourhood $N(u)$ of $u$ such that $a_{ii}^{(\ell)}(y, b) \neq 0$. Since $\Delta_{\Omega}$ is compact, there is a finite cover by such neighbourhoods and thus there is an open and dense set $V_{\Omega}$ of $b$ such
that if $b \in V_\Omega$ then $a_{ii}^{1}(u, b) \neq 0$ for all $u$ internal fixed points in $\Delta_\Omega$. If we repeat the same argument for all $\Omega \in \Xi$ we get a set
\[ V := \bigcap_{\Omega \in \Xi} V_\Omega \]
with the property that if $\{c(i_1, \ldots, i_L)\} \in V$ then $a_{ii}^{1}(u, b) \neq 0$ for all $\Omega \in \Xi$ and all internal fixed points $u \in \Delta_\Omega$. Since the collection $\Xi$ is finite, the set $V$ is open and dense.

The Claim 4.11 implies that the matrix $A_\Omega(u)$ is not nilpotent and thus not all eigenvalues of $I + A$ are equal to 1 for all $c(i_1, \ldots, i_L)$ in an open and dense set $U$. This finishes the proof of Lemma 4.10.

Let
\[ W := \bigcup_{\Omega \in \Xi} W_\Omega, \]
be the collection of all stable sets $W_\Omega$ of all internal fixed points $u \in \Delta_\Omega$, and define for $i = 1, 2, \ldots$
\[ X_i := \{ x \in \mathbb{R}^{n+} | C^i(x) \in W \} \]
be the set of points which map after $i$ iterates to a point that converges to a point in the set $Fix$. We will show that $X_i$ is a nowhere dense set and thus the set
\[ U_i := \mathbb{R}^{n+} \setminus X_i \]
is open and dense for each $i$. Then for $N$ specified by Lemma 4.7 the set
\[ G := \bigcap_{i=1}^{N} U_N \cap \Delta \]
is an open and dense set of initial conditions, which converge to either to 0 or to a vector $e$.

The major problem in showing that $X_i$ nowhere dense, is that the map $C$ is not one-to-one: in the neighbourhood of the boundary it maps multiple points to the same point on the boundary; see (A.4). We need to closely investigate the map $C(x)$ and its inverse.

### 4.4 The competition map $C(x)$ and its inverse

We investigate the inverse of the map $C$ on $\mathbb{R}^{n+}_\Omega$. We assume without loss of generality that $\Omega = (\omega^{(1)}, \ldots, \omega^{(L)})$ where $\omega^{(\ell)} = (1, 2, 3, \ldots, q(\ell))$ for all $\ell = 1, \ldots, L$. We fix $u = (u^{(1)}, \ldots, u^{(q)}) \in \mathbb{R}^{n+}$, where $u^{(\ell)} = (u_1^{(\ell)}, \ldots, u_{q(\ell)}^{(\ell)}, 0, \ldots, 0)$ with $u_i^{(\ell)} > 0$. In order to solve for (the set of) $x$ such that $u = C(x)$ we have to solve
\[
\begin{align*}
    u_i^{(\ell)} &= x_i^{(\ell)} - \kappa^{(\ell)}(1 - \frac{[L^{(\ell)}(x)]i}{K^{(\ell)}}) & \text{for } i = 1, \ldots, q(\ell) \\
    0 &\geq x_i^{(\ell)} - \kappa^{(\ell)}(1 - \frac{[L^{(\ell)}(x)]i}{K^{(\ell)}}) & \text{for } i = q(\ell) + 1, \ldots, n_\ell
\end{align*}
\]
for all $\ell = 1, \ldots, L$. This can be rewritten
\[
\begin{align*}
    u_i^{(\ell)} + \kappa^{(\ell)} &= x_i^{(\ell)} + \frac{\kappa^{(\ell)} [L^{(\ell)}(x)]i}{K^{(\ell)}} & \text{for } i = 1, \ldots, q(\ell) \\
    \kappa^{(\ell)} &\geq x_i^{(\ell)} + \frac{\kappa^{(\ell)} [L^{(\ell)}(x)]i}{K^{(\ell)}} & \text{for } i = q(\ell) + 1, \ldots, n_\ell
\end{align*}
\]
The second set of equations in (A.11) demonstrate clearly that the map $C(x)$ is not one-to-one. If the first set of the equations in (A.11) can be inverted by
\[ x^{(\ell)} = \varphi^{(\ell)}(u^{(\ell)}) \]
for all $\ell = 1, \ldots, L,$
then all solutions of (A.11) lie in the set
\[ S(u) = \{ y \in \mathbb{R}^n_+ | \ y^{(\ell)}_i = \varphi^{(\ell)}_i(u^{(\ell)}) \text{ for } i \in \omega^{(\ell)}, \ y^{(\ell)}_i \leq \kappa^{(\ell)} \text{ for } i \notin \omega^{(\ell)} \}. \]  
(A.12)
The key observation is that if \( u \in D \), a set that is nowhere dense in \( \mathbb{R}^n_+ \) for all \( \Omega \in \Xi \), then the set
\[ S(D) := \bigcup_{u \in D} S(u) \]  
(A.13)
is also nowhere dense in \( \mathbb{R}^n_+ \) for all \( \Omega \in \Xi \).

To show this we first turn our attention to invertibility of the first set of equations in (A.11)
\[ u^{(\ell)}_i + \kappa^{(\ell)} = x^{(\ell)}_i + \frac{\kappa^{(\ell)}}{K^{(\ell)}} [L^{(\ell)}(x)]_i \text{ for } i = 1, \ldots, q(\ell), \ell = 1, \ldots, L. \]  
(A.14)
We note that in (A.14) the expression \( L^{(\ell)}(x) \) involves \( x^{(\ell)}_{q(\ell)+1}, \ldots, x^{(\ell)}_{n_\ell} \). If we restrict our search to \( x \in \mathbb{R}^n_+ \), i.e. to those \( x \) with \( x^{(\ell)}_i = 0 \) for all \( i > q(\ell) \), then the set of equations (A.14) defines a function
\[ C_{\Omega} : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \]  
(A.15)
by
\[ u^{(\ell)}_i = C_{\Omega}^{(\ell)}(x) = X^{(\ell)}_i + \kappa^{(\ell)} \left( \frac{[L^{(\ell)}(X)]_i}{K^{(\ell)}} - 1 \right), \]
where \( X := (X^{(1)}, \ldots, X^{(L)}) \) and \( X^{(\ell)} = (x^{(\ell)}_1, \ldots, x^{(\ell)}_{q(\ell)}, 0, 0, \ldots, 0) \).

**Lemma 4.13** If \( \kappa^{(\ell)} \leq \frac{c_{\Omega}}{c_{\Omega}} \) for all \( \ell \), then the maps \( C_{\Omega}(x) \) are invertible as functions from \( \mathbb{R}^n_+ \) to \( \mathbb{R}^n_+ \) for all \( \Omega \in \Xi \).

**Proof.** We fix \( \Omega \in \Xi \). We have computed the derivative matrix \( \frac{dC_{\Omega}}{dx}(x) \) in (A.8) and its components in (A.10). Since all values of \( x^{(\ell)}_i \leq 1 \) we can estimate
\[ [A(x)]^{(\ell)}_{i,s} \leq \kappa^{(\ell)} \frac{c_{\Omega}}{c_{\Omega}} \leq 1. \]
Therefore
\[ \det(\frac{dC_{\Omega}}{dx}(x)) = \det(I + A(x)) \neq 0. \]
\[ \square \]

Let \( \varphi_{\Omega} : \mathbb{R}^n_+ \to \mathbb{R}^n_+ \) denote the inverse of \( C_{\Omega} \). The following Theorem addresses the non-uniqueness of the inverse to \( C(x) \).

**Theorem 4.14** Let \( D \subset \mathbb{R}^n_+ \) be a nowhere dense closed set in \( \mathbb{R}^n_+ \) such that \( D \cap \mathbb{R}^n_+ \) is nowhere dense and closed in \( \mathbb{R}^n_+ \) for all \( \Omega \in \Xi \). Then the set
\[ C^{-1}(D) := \{ x \in \mathbb{R}^n_+ | C(x) \in D \} \]  
is nowhere dense and closed in \( \mathbb{R}^n_+ \) and \( C^{-1}(D) \cap \mathbb{R}^n_+ \) is nowhere dense and closed in \( \mathbb{R}^n_+ \) for all \( \Omega \in \Xi \).

**Proof.** Observe that the set \( C^{-1}(D) = \bigcup_{u \in D} S(u) \) (compare (A.13)), where \( S(u) \) has the form (A.12). Therefore \( C^{-1}(D) \) is homotopic to \( \varphi_{\Omega}(D) \times \prod_{\ell=1}^{n_{\ell}} [0, \kappa^{(\ell)}) \). Since \( \varphi_{\Omega} \) is a continuous function and \( D \) is closed, \( C^{-1}(D) \) is closed. Now \( \mathbb{R}^n_+ \) is closed for every \( \Omega \) and so \( C^{-1}(D) \) is closed in \( \mathbb{R}^n_+ \).

Now we show that the complement of \( C^{-1}(D) \) in \( \mathbb{R}^n_+ \) is dense in \( \mathbb{R}^n_+ \). Select an arbitrary point \( x_0 \in C^{-1}(D) \) in \( \mathbb{R}^n_+ \) for some \( \Omega \). We allow \( \Omega = (\omega^1, \ldots, \omega^d) \) with \( \omega^{(\ell)} = \{1, \ldots, n_{\ell}\} \) for all \( m \), in which case all components of \( x_0 \) are nonzero. Take an \( \epsilon \)-neighbourhood \( N_{\epsilon} \subset \mathbb{R}^n_+ \) of \( x_0 \). To show that \( C^{-1}(D) \) is
nowhere dense in $\mathbb{R}_n^{\Omega}$ we need to find a point $x \in N_\epsilon$ which does not belong to $C^{-1}(D)$. Since we restrict to $x \in \mathbb{R}_n^{\Omega}$ we have

$$C^{-1}(D) \cap \mathbb{R}_n^{\Omega} = \varphi_\Omega(D).$$

Take $y \in \varphi_\Omega(D) \cap N_\epsilon$ and choose $\delta$ such that a $N_\delta$ neighbourhood of $y$ lies in $N_\epsilon$. Since $\varphi_\Omega$ is a $C^1$ function with non-singular derivative at $z := C(y)$ by the Inverse Mapping Theorem, $\varphi_\Omega^{-1}(N_\delta)$ is an open neighbourhood of the point $z := C(y), z \in D$. Since $D$ is nowhere dense in $\mathbb{R}_n^{\Omega}$, there is a point $w \in \mathbb{R}_n^{\Omega}$ in this image with $w \notin D$. Then there is $x \in \varphi_\Omega(w) \in N_\delta \subset N_\epsilon$ for which we have $x \notin C^{-1}(D) \cap N_\epsilon$. Since $\epsilon$ was arbitrary, the complement of $C^{-1}(D)$ is dense in $\mathbb{R}_n^{\Omega}$.

If $\Omega \subset \Omega'$ and $C^{-1}(D)$ is nowhere dense in $\mathbb{R}_n^{\Omega'}$ then clearly $C^{-1}(D)$ is nowhere dense in $\mathbb{R}_n^{\Omega}$. 

\[\square\]

### 4.5 Proof of Theorem 4.1.

To prove Theorem 4.1, we use the Theorem 4.14 with $D := B$. Recall that

$$X_i := \{x \in \mathbb{R}_n^{\Omega} \mid C^i(x) \in W\}, \quad i = 1, 2, \ldots.$$ 

Then by Lemma 4.10 the set $W$ is closed and nowhere dense in $\mathbb{R}_n^{\Omega}$ and $W \cap \mathbb{R}_n^{\Omega}$ is nowhere dense closed in $\mathbb{R}_n^{\Omega}$ for all $\Omega \in \Xi$.

We proceed by induction, where Theorem 4.14 provides the induction step. Assume $X_i$ is closed and nowhere dense set in $\mathbb{R}_n^{\Omega}$ such that $X_i \cap \mathbb{R}_n^{\Omega}$ is nowhere dense closed in $\mathbb{R}_n^{\Omega}$ for $\Omega \in \Xi$. Then by Theorem 4.14 $X_{i+1}$ is closed and nowhere dense in $\mathbb{R}_n^{\Omega}$, and $X_{i+1} \cap \mathbb{R}_n^{\Omega}$ is closed and nowhere dense for all $\Omega \in \Xi$.

By induction we conclude that $X_i$ is closed and nowhere dense in $\mathbb{R}_n^{\Omega}$ and $X_i \cap \mathbb{R}_n^{\Omega}$ is closed and nowhere dense in $\mathbb{R}_n^{\Omega}$ for all $\Omega \in \Xi$. Therefore the set $U_i = \mathbb{R}_n^{\Omega} \setminus X_i$ is open and dense for all $i$. The set $G = \bigcap_{i=1}^N U_N \cap \Delta$ where $N$ is selected by Lemma 4.7 is an open and dense set in $\Delta$ and $G \cap \Delta_\Omega$ is open and dense in in $\Delta_\Omega$ for any $\Omega \in \Xi$. The set $G$ represents a set of initial conditions whose iterations will never enter the set $W$, and therefore do not converge to any fixed point in the set of internal fixed points $Fix$. By Lemma 4.7 the corresponding trajectory then converges to either 0 vector or to a vector $e = (a_1 e^{(1)}_i, \ldots, a_L e^{(L)}_i)$ for some positive collection of $a_i$ and for some choice of vectors $e_i$.

\[\square\]

### References


