NPB 163/PSC 128 lecture notes
Fourier Analysis

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Why frequency analysis?

- Many signals in the natural environment are conveniently described in terms of a superposition of wobbly functions (e.g., many sounds are produced by vibrating membranes).

- Sines and cosines are eigenfunctions of linear, time-invariant systems. Thus, they can be used to conveniently characterize the response of a linear system. Also, convolution, which is a complicated signal transformation in the time or space domain, is performed by simple multiplication in the frequency domain.

The Fourier series

- Joseph Fourier's theorem, in its most general form, states that any function may be described in terms of a superposition of odd and even functions. Specifically, Fourier proposed that a signal $s(t)$ may be decomposed into a summation of sinewaves (or cosine-waves) of different frequencies, amplitudes and phases:

$$s(t) = \sum_{f=0}^{\infty} A(f) \cos(2\pi\, f\, t + \phi(t))$$

- What is remarkable about this is that $s(t)$ can be anything—the waveform produced by a bird chirping, the sound of your dishwasher, electromagnetic waves, etc.

- The amplitudes tell you how much of each frequency is present in the signal. For example, a pure tone (e.g., the waveform emitted by a tuning fork) would have equal to zero for all frequencies except for one. The sound produced when you say "shhh" would have amplitudes distributed across many frequencies.

- The Fourier series tells us that a sound may be decomposed in terms of sinewaves, but it doesn't tell us how to do it - i.e., it doesn't tell us what amplitudes A to assign to each f. For this we need the Fourier transform.

The Fourier transform

- The Fourier transform basically provides a way of representing a signal in a different space—i.e., in the frequency domain. You put into the Fourier transform a function of time or space, $s(t)$, and you get out a function of frequency, $S(f)$. The Fourier transform is formally defined as follows:

$$S(f) = \int s(t)\, e^{-i\,2\pi\,f\,t}$$

- Thus, the Fourier transform is essentially the inner-product of the signal $s(t)$ with the complex exponential $e^{-i\,2\pi\,f\,t}$, evaluated at different values of f.

- The complex exponential is just a real cosine-wave and an imaginary sine-wave:

$$e^{-i\,2\pi\,f\,t} = \cos(2\pi\,f\,t) + i\,\sin(2\pi\,f\,t)$$

  Thus, the Fourier integral may be written alternatively as

$$S(f) = \int s(t)\,\cos(2\pi\,f\,t) + i\int s(t)\,\sin(2\pi\,f\,t)$$

- Here we can see that $S(f)$ is a complex number. The real part tells us the result of multiplying our signal together with a cosine-wave, the imaginary part tells us the result of multiplying the signal together with a sine-wave.

- The amplitude A of each frequency component contained in is given by the *modulus* of S, which is defined as

$$A(f) = |S(f)| \equiv \sqrt{\Re\{S(f)\}^2 + \Im\{S(f)\}^2}$$

  where $\Re$ and $\Im$ denote the "real part" and "imaginary part," respectively.

- The phase ϕ can be extracted from the ratio of the real an imaginary components of as follows:

$$\phi(f) = \tan^{-1}\frac{\Im\{S(f)\}}{\Re\{S(f)\}}$$

Convolution theorem

- The convolution of two functions,

$$y(t) = h(t) * x(t)$$

  may be performed in the frequency domain via multiplication:

$$Y(f) = H(f)\,X(f)$$

  where $Y(f)$, $H(f)$, and $X(f)$ are Fourier transforms of $y(t)$, $x(t)$, and $h(t)$ respectively.

- That this is so is due to the fact that 1) sines and cosines are eigenfunctions of linear time-invariant systems, and 2) any function may be represented as a sum of sines and cosines.